Multiple scattering of sound by correlated monolayers[a)]

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Earlier results for scattering by random uncorrelated planar distributions, and by doubly periodic planar configurations of relatively arbitrary obstacles, are generalized to pair-correlated nonsymmetrical monolayers. The existing development for parallel cylinders in terms of the Zernike–Prins one-dimensional pair function $p(x)$, is extended to analogous two-dimensional distributions specified by $p(\mathbf{R})$ for aligned impenetrable disks. We obtain the average multiple scattered transmitted and reflected waves, and an energy conserving approximation of the differential scattering cross section per unit area. Simplified forms are developed to facilitate determing p by inverting measured data. Closed form low-frequency results are derived for identical ellipsoids aligned nonsymmetrically to the plane of centers, and the array multipole-coupling processes are discussed in terms of functions of p and their approximations.

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INTRODUCTION

In previous papers[1–5] we considered scattering of waves by random and by periodic planar distributions of relatively arbitrary obstacles. The developments emphasized scatterer shapes symmetrical to the plane of centers, e.g., radially symmetric obstacles, parallel elliptic cylinders, or ellipsoids with a principal semidiameter perpendicular to the plane of centers, or more generally, a monolayer of obstacles with reflection symmetry in a midplane (henceforth a symmetrical monolayer). For uncorrelated random distributions, in the course of development of a model for rough surfaces,[1] we obtained the average multiple scattered transmitted and reflected waves, and an energy conserving approximation for the differential scattering cross section per unit area. For periodic monolayers (gratings[3] of equally spaced parallel cylinders in two dimensions and the doubly periodic planar array[4] of bounded obstacles in three dimensions), we derived the multiple scattered solutions for arbitrary ratios of spacings (b) to wavelength (λ). The results for small b/λ were also relevant for random distributions for negligible incoherent scattering. In addition, we considered correlated monolayers of parallel cylinders[5] in terms of the Zernike–Prins pair distribution function[6] $p(x)$, with emphasis on reduction procedures leading to the results for the uncorrelated[1] and periodic[3] limits.

The uncorrelated limit corresponds to a one-dimensional sparse gas in statistical mechanics, with minimum separation (b) of particle centers small compared to average separation ($\bar{b}$); the periodic limit ($b \to \bar{b}$) corresponds to a one-dimensional crystal. Using transform methods we converted $p(x)$ to a residue series[5] to obtain more rapidly converging forms for $b < \bar{b}$ and $b \sim \bar{b}$. For cylinders of width $2a$ in the plane of centers, we considered $2a < b$ and arbitrary $kb = 2\pi b/\lambda$, in order to demonstrate that correlation effects determined by $b/\bar{b} = \rho b = w$ (the statistical mechanics packing factor) accounted for the full range of phenomena from $w \approx 0$ for the sparse gas on to $w \to 1$ for the deterministic periodic limit.

Now we generalize the earlier results for cylinders to multiple scattering by correlated nonsymmetrical monolayers of bounded obstacles, and emphasize reduction procedures for data inversion purposes. The initial development for cylinders,[5] based on separation of variables (1953), was simplified by a Green's function procedure (1959); the present procedure for bounded obstacles (as well as for cylinders) represents a further simplification. We specialize the general representation[7] for the solution Ψ in terms of the multiple scattering amplitude G of an obstacle in an arbitrary configuration, with G related functionally to the amplitude g in isolation, and then average. We consider the average field $\langle \Psi \rangle$ for a statistically homogeneous ensemble of configurations of aligned identical obstacles with one-particle statistics specified by the average number ρ of particles in unit area, and two-particle statistics determined by $\rho p(\mathbf{R})$. For p, the distribution function for the separation $\mathbf{R}$ of centers of pairs, we use general statistical mechanics results[8,9] for impenetrable aligned identical disks $\mathbf{b}(\hat{\mathbf{R}})/2$, such that the exclusion curve $\mathbf{b}$ (the curve around one center that excludes all others) determines $p(\mathbf{R})$. In general, we assume that $\mathbf{b}$ and p have the same inversion and reflection symmetrics as an ellipse.

Although even the radially symmetric special case $p(\mathbf{R})$, the radial distribution function for identical circular disks is not known explicitly, the results derived for $\langle \Psi \rangle$ and $\langle |\Psi|^2 \rangle$ are reduced to facilitate determining $p(\mathbf{R})$ by inverting measured data, essentially as in the corresponding x-ray problems. For the direct problem of predicting $\langle \Psi \rangle$ for a particular distribution, or of synthesizing a monolayer with special characteristics, several procedures for approximating p exist for special ranges of the parameters. The leading terms of the virial expansion of p suffice for sparse concentrations, and for arbitrary concentration and low frequencies (small kb) a required integral of p is available as a simple function of the packing density in the plane.[10] More generally, $p(R)$ can be obtained numerically from the Percus–Yevick or other approximations[8,9] for the pair function. For identical aligned ellipsoidal particles that intersect the plane of centers as an ellipse $\mathbf{a}(\hat{\mathbf{R}})$, the exclusion curve is the symmetrical ellipse

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$2\mathbf{a}(\hat{\mathbf{R}})$. For greater generality (essentially as for the corresponding volume distributions[10]) we take the exclusion curve $\mathbf{b}(\hat{\mathbf{R}})$ as neither similar nor similarly aligned with $\mathbf{a}(\hat{\mathbf{R}})$.

In the following, for brevity, we use (1:9) for Eq. (9) of Ref. 1, etc. Section I introduces required notation and representations in the form of a brief sketch of the single and many body scattering problems, and Sec. II derives $\langle\Psi\rangle$ for an incident plane wave ϕ, and the functional equation for the average multiple scattered amplitude $G[g]$. Section III considers the average energy functions, and provides physical motivation for Sec. IV in which G is expressed in terms of a transform of g that delineates the loss mechanisms in the array, and that leads directly to simplified results for data inversion. For the nonsymmetrical monolayer, for incident waves ϕ and ϕ' that are images in the plane of centers, we develop explicit approximations in terms of p and g for the magnitudes and phases of the transmission and reflection coefficients and for the differential scattering cross section per unit area. Section V uses Fourier series in $G[g]$ to obtain algebraic systems for the corresponding multiple scattered multipole coefficients $A[a]$ in terms of the distribution integrals $\mathscr{H}$ that characterize the array (i.e., in terms of continuium analogs of the lattice sums for the periodic cases).[3,4] We consider $\mathscr{H}$ in terms of p (virial expansions and Fourier transforms), and develop low-frequency approximations. Section VI derives closed form results for small ellipsoids aligned nonsymmetrically to the plane of centers, with emphasis on multipole coupling effects. These explicit results provide illustrations of the general considerations of Sec. IV for ϕ and ϕ' incident on nonsymmetrical monolayers. For small kb, except for sparse packings, wavelength-independent multipole coupling effects may be misinterpreted as changes in the shape of the obstacles; e.g., because of array coupling, a spherical obstacle corresponds to an equivalent ellipsoid flattened along the array normal and broadened in the plane of centers.

I. REPRESENTATIONS

Consider a plane wave $\phi e^{-i\omega t}$ incident on an obstacle (in two or three dimensions) with center at $r=0$, the center of its smallest circumscribing circle or sphere (of radius a). We take

$$
\begin{aligned}
&\phi = e^{i\mathbf{k}\cdot\mathbf{r}};\ \mathbf{r} = r\hat{\mathbf{r}},\quad r^2 = z^2 + R^2\\
&\hat{\mathbf{r}} = \hat{\mathbf{r}}(\theta,\varphi) = \hat{\mathbf{z}}\cos\theta + \hat{\mathbf{R}}(\varphi)\sin\theta,\\
&\hat{\mathbf{R}}(\varphi) = \hat{\mathbf{x}}\cos\varphi + \hat{\mathbf{y}}\sin\varphi;\\
&\mathbf{k} = k\hat{\mathbf{k}},\ k = 2\pi/\lambda,\\
&\hat{\mathbf{k}} = \hat{\mathbf{r}}(\theta_0,\varphi_0) = \hat{\mathbf{r}}_0 = \hat{\mathbf{z}}\cos\theta_0 + \hat{\mathbf{R}}_0\sin\theta_0,\\
&\hat{\mathbf{R}}_0 = \hat{\mathbf{R}}(\varphi_0).
\end{aligned}
\tag{1}
$$

When convenient, we work with direction cosines

$$
\hat{\mathbf{r}} = \hat{\mathbf{r}}\gamma + \hat{\mathbf{x}}\alpha + \hat{\mathbf{y}}\beta = \textstyle\sum \hat{\mathbf{x}}_i\gamma_i,\quad \gamma = \gamma_1 = \cos\theta,
$$

$$
\alpha = \gamma_2 = \sin\theta\cos\varphi.
$$

$$
\beta = \gamma_3 = \sin\theta\sin\varphi.
$$

For cylinders with axes along $\hat{\mathbf{y}}$, we set $\varphi = 0, \beta = \gamma_3 = 0$. In general, we use terminology and procedures that apply for three dimensions and can be specialized to two, and treat two- and three-dimensional problems in parallel; if two forms of a function or operator arise, we list the cylindrical first.

In the region external to the scatterer (outside the volume $\mathscr{V}$ bounded by $\mathscr{S}$), the field $\psi = \phi + u$ satisfies $(\nabla^2 + k^2)\psi = 0$ with u as a radiative function[7]

$$
u(\mathbf{r}) = c_0\int[h_0(k|\mathbf{r}-\mathbf{r}'|)\partial_n u(\mathbf{r}') - u\partial_n h_0]\,d\mathscr{S}(\mathbf{r}') \equiv \{h_0,u\};
$$

$$
h_0(x) = H_0^{(1)}(x),\quad h_0^{(1)}(x);\quad c_0 = 1/i4,\ k/i4\pi.
\tag{2}
$$

Here, $\mathscr{S}$ is the obstacle's surface (or any surface that isolates it from the field point $\mathbf{r}$), and ∂_n is the outward normal derivative. For $r\sim\infty$,

$$
u \sim h(kr)g(\hat{\mathbf{r}},\hat{\mathbf{k}}),\quad g(\hat{\mathbf{r}},\hat{\mathbf{k}}) = \{e^{-i\mathbf{k}_r\cdot\mathbf{r}'},u\},\quad \mathbf{k}_r = k\hat{\mathbf{r}},
\tag{3}
$$

with h as the asymptotic form of h_0, and $g(\hat{\mathbf{r}},\hat{\mathbf{k}})$ as the scattering amplitude. Substituting the plane-wave spectral representation for h_0 in (2) we obtain, at least for $r>a$ for all $\hat{\mathbf{r}}$,

$$
\begin{aligned}
&u(\hat{\mathbf{r}}) = \int_c e^{i\mathbf{k}_c\cdot\mathbf{r}} g(\hat{\mathbf{r}}_c,\hat{\mathbf{k}}),\\
&\mathbf{k}_c = k\hat{\mathbf{r}}_c,\quad \hat{\mathbf{r}}_c = \hat{\mathbf{r}}(\theta_c,\varphi_c),
\end{aligned}
\tag{4}
$$

where $\int_c$ equals $(1/\pi)\int d\theta_c$ with contour as for $H_0^{(1)}$, or

$$
\left(\frac{1}{2\pi}\right)\int d\Omega\,(\theta_c,\varphi_c)
$$

with contours as for $h_0^{(1)}$. Decomposing g as Fourier series (trigonometric or spherical harmonic) of scattering coefficients $a_n(\hat{\mathbf{r}}_0)$ or $a_n^m(\hat{\mathbf{r}}_0)$, we obtain[7] from (4) for u the corresponding series in special functions $H_n^{(1)}(kr)e^{in\theta}$ or $h_n^{(1)}(kr)Y_n^m(\hat{\mathbf{r}})$. We reserve such series for special computations, and use primarily (2) and (4) to derive general results.

The integral in (4) corresponds to h_0. In terms of the associated integral operator for $j_0 = \mathrm{Re}\,h_0$, the mean $(\mathscr{M})$ over all real directions of observation, we write the energy theorem as

$$
\begin{aligned}
&-\sigma_0\,\mathrm{Re}\,g(\hat{\mathbf{k}},\hat{\mathbf{k}}) = \sigma_a + \sigma_s;\\
&\sigma_a = -\tfrac{1}{2}\sigma_0\{\psi^*,\psi\},\quad \sigma_s = \tfrac{1}{2}\sigma_0\{u^*,u\} = \sigma_0\mathscr{M}|g(\hat{\mathbf{r}},\hat{\mathbf{k}})|^2;\\
&\sigma_0 = \frac{4}{k},\ \frac{4\pi}{k^2};\\
&\mathscr{M} = \frac{1}{2\pi}\int_0^{2\pi}d\theta,\ \frac{1}{4\pi}\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\sin\theta,
\end{aligned}
\tag{5}
$$

where σ_a and σ_s are the absorption and scattering cross sections. This theorem follows directly from the definition of σ_a in terms of $\psi = \phi + u$ by using Green's theorem and (3). The same procedure applied to $\psi(\hat{\mathbf{r}}_1)$ and $\psi(\hat{\mathbf{r}}_2)$, the solutions for two different directions of incidence $\hat{\mathbf{r}}_1,\hat{\mathbf{r}}_2$ (each solution satisfying the same conditions on $\mathscr{S}$ and in $\mathscr{V}$), gives the usual reciprocity relation

$$
\begin{aligned}
&\{\psi(\hat{\mathbf{r}}_1),\psi(\hat{\mathbf{r}}_2)\} = 0:\quad g(-\hat{\mathbf{r}}_1,\hat{\mathbf{r}}_2) = g(-\hat{\mathbf{r}}_2,\hat{\mathbf{r}}_1),\\
&g(\hat{\mathbf{r}},\hat{\mathbf{k}}) = g(-\hat{\mathbf{k}},-\hat{\mathbf{r}}).
\end{aligned}
\tag{6}
$$

The following development applies for general obstacles and all conditions on $\mathscr{S}$ and in $\mathscr{V}$ for which (6) holds.

For a fixed planar configuration of N obstacles with centers $(\hat{\mathbf{R}}_s)$ in the xy plane, we write the solution external to all obstacles as

$$\begin{aligned}
\Psi &= \phi + \sum_{s=1}^{N} U_s(\mathbf{r}-\mathbf{R}_s;\mathbf{R}_1,\mathbf{R}_2,\ldots,\mathbf{R}_N)e^{i\mathbf{k}\cdot\mathbf{R}_s} = \phi + \mathscr{U},\\
\mathbf{R}_s &= x_s\hat{\mathbf{x}} + y_s\hat{\mathbf{y}} = R_s\hat{\mathbf{R}}(\varphi_s),\\
U_s(\mathbf{r}-\mathbf{R}_s) &= \int_c e^{i\mathbf{k}_c\cdot(\mathbf{r}-\mathbf{R}_s)}G_s(\hat{\mathbf{r}}_c,\hat{\mathbf{k}}),\\
G_s(\hat{r},\hat{k}) &= \{e^{-i\mathbf{k}_r\cdot\mathbf{r}'},U_s\}_s,
\end{aligned} \tag{7}$$

where the multiple scattering amplitude G_s is the form (3) over $\mathscr{S}_s(\mathbf{r}')$, and $\mathbf{r}'$ is the local vector from $\mathbf{R}_s$. For $|\mathbf{r}-\mathbf{R}_s|\sim\infty$, we have $U_s\sim h(k|\mathbf{r}-\mathbf{R}_s|)G_s$. With reference to the center of scatterer t, we display the phase factor introduced by $\phi(\mathbf{R}_t)$ and use

$$\Psi = \Psi_t e^{i\mathbf{k}\cdot\mathbf{R}_t} = (\Phi_t + U_t)e^{i\mathbf{k}\cdot\mathbf{R}_t},$$

$$\Phi_t = e^{i\mathbf{k}\cdot\mathbf{r}'} + \Sigma_s' U_s(\mathbf{R}_t + \mathbf{r}' - \mathbf{R}_s)e^{i\mathbf{k}\cdot(\mathbf{R}_s - \mathbf{R}_t)},$$

where Σ_s' is the sum over $s\neq t$. We require that Ψ_t, Φ_t, U_t satisfy the same conditions on $\mathscr{S}_t$ and $\mathscr{V}_t$ as ψ,ϕ,u for the corresponding scatterer in isolation, and that $(\nabla^2 + k^2)U_t = 0$ external to $\mathscr{V}_t$. Consequently from $\{\Psi(\hat{\mathbf{r}}_1),\Psi(\hat{\mathbf{r}}_2)\}_t = 0$, we have

$$G_t(-\hat{\mathbf{r}}_1,\hat{\mathbf{r}}_2) = \{\phi(\hat{\mathbf{r}}_1),U_t(\hat{\mathbf{r}}_2)\}_t = \{\Phi_t(\hat{\mathbf{r}}_2),u(\hat{\mathbf{r}}_1)\}_t;$$

substituting for Φ_t with U_s as in (7), and using the definitions of g and G, it follows that

$$G_t(\hat{\mathbf{r}},\hat{\mathbf{k}}) = g_t(\hat{\mathbf{r}},\hat{\mathbf{k}}) + \Sigma_s' e^{-i\mathbf{k}\cdot\mathbf{R}_{ts}}\int_c e^{i\mathbf{k}_c\cdot\mathbf{R}_{ts}} g_t(\hat{\mathbf{r}},\hat{\mathbf{r}}_c)G_s(\hat{\mathbf{r}}_c,\hat{\mathbf{k}}),\mathbf{R}_{ts} = \mathbf{r}_t - \mathbf{r}_s, \tag{8}$$

at least if the scatterers's projections do not overlap on $\hat{\mathbf{R}}_{ts} = \mathbf{R}_{ts}/R_{ts}$. The functional equation (8) is essentially a reciprocity relation between the multiple (G) and single (g) scattering amplitudes, for both the direct and inverse scattering problems. We emphasize the direct problem $G[g]$ as if g were known, but the results also apply for the inverse $g[G]$.

II. THE AVERAGE WAVE

We average Ψ over a statistically homogeneous ensemble of planar configurations of identical aligned obstacles with one-particle statistics specified by the average number (ρ) of centers per unit area. Thus

$$\langle\Psi\rangle = \phi + \rho\int e^{i\mathbf{k}\cdot\mathbf{R}_s}\langle U_s(\mathbf{r}-\mathbf{R}_s)\rangle_s = \phi + \langle\mathscr{U}\rangle, \tag{9}$$

where $\langle U_s\rangle_s$, the average with one variable held fixed, depends only on $\mathbf{R}_s$ (now a dummy). From (7),

$$\begin{aligned}
\langle U_s(\hat{\mathbf{r}}-\mathbf{R}_s)\rangle_s &= \int_c e^{i\mathbf{k}_c\cdot(\mathbf{r}-\mathbf{R}_s)}\langle G_s(\hat{\mathbf{r}}_c,\hat{\mathbf{k}})\rangle_s\\
&= \int_c e^{ik_c\cdot(\mathbf{r}-\mathbf{R}_s)}G(\hat{\mathbf{r}}_c,\hat{\mathbf{k}}),
\end{aligned} \tag{10}$$

with $\langle G_s\rangle_s = G$ independent of $\mathbf{R}_s$, because $e^{i\mathbf{k}\cdot\mathbf{R}_s}$ was factored at the start.

Substituting (10) into (9), we write $\mathbf{R}_s = \xi\hat{\mathbf{x}} + \eta\hat{\mathbf{y}}$, $d\mathbf{R}_s = d\xi\, d\eta$, and $(\hat{\mathbf{r}}_0 - \hat{\mathbf{r}}_c)\cdot\mathbf{R} = (\alpha_0 - \alpha_c)\xi + (\beta_0 - \beta_c)\eta$; the integral over ξ reduces to

$$\int_{-\infty}^{\infty} d\xi\, e^{ik(\alpha_0 - \alpha_c)\xi} = \frac{2\pi\delta(\alpha_0 - \alpha_c)}{k},$$

and that over η to $2\pi\delta(\beta_0 - \beta_c)/k$. The δ functions correspond to $\hat{\mathbf{r}}_c = \hat{\mathbf{r}}(\theta_0,\varphi_0), \hat{\mathbf{r}}(\pi - \theta_0,\varphi_0) = \hat{\mathbf{r}}_0,\hat{\mathbf{r}}_0' = \hat{\mathbf{k}},\hat{\mathbf{k}}'$ for $z \gtrless 0$. Transforming $\int d\Omega_c$ to $\int\int d\alpha_c\, d\beta_c\, \gamma_c^{-1}$ in terms of real variables α_c,β_c [with $\gamma_c = (1-\alpha_c^2-\beta_c^2)^{1/2} = |\gamma_c|$ or $i|\gamma_c|$ for $\alpha^2+\beta^2<1$ or >1], we reduce the average scattered wave $\langle\mathscr{U}\rangle$ as before[1] to the specular values

$$\begin{aligned}
\langle\mathscr{U}_>\rangle &= \phi\, 2CG(\hat{\mathbf{k}},\hat{\mathbf{k}}) \equiv \phi\, 2CG_{00},\\
\langle\mathscr{U}_<\rangle &= \phi'\, 2CG(\hat{\mathbf{k}}',\hat{\mathbf{k}}) = \phi'\, 2CG_{0'0};\\
C &= \rho/k\gamma_0,\quad \rho\pi/k^2\gamma_0;\quad \phi' = e^{i\mathbf{k}'\cdot\mathbf{r}},\\
\hat{\mathbf{k}}' &= \hat{\mathbf{r}}_0' = \hat{\mathbf{r}}(\pi - \theta_0,\varphi_0) = \hat{\mathbf{k}} - 2\hat{\mathbf{k}}\cdot\hat{\mathbf{z}}\hat{\mathbf{z}},
\end{aligned} \tag{11}$$

where ϕ' is the image of ϕ in the plane $z=0$. The subscripts ($>$ and $<$) indicate that the forms hold at least for $z>a$ and $z<-a$, respectively. (Were the incident wave ϕ' instead of ϕ then $\hat{\mathbf{k}}$ and $\hat{\mathbf{k}}'$ would be interchanged.) The corresponding net average fields in the transmitted and reflected regions are

$$\begin{aligned}
\langle\Psi_>\rangle &= \phi(1+2CG_{00}) = \phi T,\\
\langle\Psi_<\rangle &= \phi + \phi'\,2CG_{0'0} = \phi + \phi' R,
\end{aligned} \tag{12}$$

with T and R as the coherent transmission and reflection coefficients.

The two values in (11) correspond to one wave form multiplied by the forward scattered (G_{00}) or reflected ($G_{0'0}$) amplitudes:

$$\langle\mathscr{U}\rangle = e^{\pm ikz\gamma_0 + i\mathbf{k}\cdot\mathbf{R}}2CG(\pm\hat{\mathbf{z}}\cos\theta_0 + \hat{\mathbf{R}}_0\sin\theta_0,\hat{\mathbf{k}}_0) \propto e^{ik|z|\gamma_0 + i\mathbf{k}\cdot\mathbf{R}}$$

is a wave outgoing from $z=0$. The images ϕ and ϕ' have the same $|z|,x,y$ dependence and comprise the central propagating mode of the periodic cases[3,4] discussed earlier; the earlier results for the case of one propagating mode, $kb(1\pm\sin\theta_0)<2\pi$, provide bounds for present low-frequency approximations.

Similarly, $\langle G_t\rangle_t = G$ is obtained from the ensemble average of (8) for pair-correlated scatterers with separation ($\mathbf{R}_{ts}$) specified by $\rho p(_{ts})$, such that $p(\mathbf{R})\sim 1$ for $R\sim\infty$, and $p(\mathbf{R}) = 0$ for $R<b(\hat{\mathbf{R}})$ with $b(\hat{\mathbf{R}})$ as the minimum separation of pairs. We assume that $p(\mathbf{R})$ is fully determined by the curve $\mathbf{R} = \mathbf{b}(\hat{\mathbf{R}})$, the exclusion curve enclosing the area containing the center of only one scatterer. For radially symmetric statistics, $b(\hat{\mathbf{R}}) = b$ is constant and $p = p(R)$ is independent of direction. More generally, we consider exclusion areas having the inversion and reflection symmetries of an ellipse, henceforth elliptical symmetry. From (8),

$$\begin{aligned}
\langle G_t(\hat{\mathbf{r}},\hat{\mathbf{k}})\rangle_t &= g(\hat{\mathbf{r}},\hat{\mathbf{k}}) + \rho\int d\mathbf{R}_s\, p(\mathbf{R}_{ts})e^{-i\mathbf{k}\cdot\mathbf{R}_{ts}}\\
&\quad\times\int_c g(\hat{\mathbf{r}},\hat{\mathbf{r}}_c)\langle G_s(\hat{\mathbf{r}}_c,\hat{\mathbf{k}})\rangle_{st}e^{i\mathbf{k}_c\cdot\mathbf{R}_{ts}},
\end{aligned} \tag{13}$$

where $\langle G_s\rangle_{st}$ is the average over all variables but $\mathbf{R}_s$ and $\mathbf{R}_t$. To obtain a deterministic equation for $\langle G_t\rangle_t = G$, we pro-

ceed as before[1,5] and replace $\langle G_s \rangle_{st}$ by $\langle G_s \rangle_s = G$. See discussion after (10:16) for citations to work of Lax on the corresponding approximation for random media, and for alternatives by Keller and by the writer.

We consider

$$g(\hat{\mathbf{r}},\hat{\mathbf{k}}) + \rho \int d\,\mathbf{R}\, p(\mathbf{R}) e^{-i\mathbf{k}\cdot\mathbf{R}} \int_c e^{i\mathbf{k}_c\cdot(\mathbf{z}-\mathbf{R})} g(\hat{\mathbf{r}}.\hat{\mathbf{r}}_c) G(\hat{\mathbf{r}}_c,\hat{\mathbf{k}});$$

$$\int_c = \int_r + \int_e \tag{14}$$

$$= \frac{1}{\pi}\int_{-\pi/2+i\infty}^{\pi/2-i\infty} d\theta_c,\ \ \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi_c \int_0^{\pi/2-i\infty} d\theta_c \sin\theta_c,$$

in terms of simple contours, and $\mathbf{z} = \hat{\mathbf{z}}\epsilon$ to provide a convergence factor with exponent $ikz\cos\theta_c = k|\epsilon|\sinh\tau$ for $\theta_c = \pm(\pi/2 - i\tau)$ and $\tau \sim \infty$. We asume that $G(\hat{\mathbf{r}},\hat{\mathbf{k}})$ may be equated with the limit of (14) for $z = |\epsilon| \to 0$, or for $z = -|\epsilon| \to 0$ and $\hat{\mathbf{r}}_c$ replaced by its image $\hat{\mathbf{r}}_c'$; in general, we work with the mean of the two which makes the symmetry of the problem more evident. The form $\mathscr{F}(\hat{\mathbf{r}}_c) = \mathscr{G}(\hat{\mathbf{r}}_c) + \mathscr{G}(\hat{\mathbf{r}}_c')$, with $\mathscr{G}(\hat{\mathbf{r}}_c) = g(\hat{\mathbf{r}},\hat{\mathbf{r}}_c)G(\hat{\mathbf{r}}_c\hat{\mathbf{k}})$, depends only on the real variables φ_c and $\sin\theta_c$ (i.e., $\sin\theta_c = \mp\cosh\tau$ is real even for $\theta_c = \pm\pi/2 \mp i\tau$) as may be seen by using trigonometric or spherical harmonic series representations. For cylinders, $\mathscr{F}(\hat{\mathbf{r}})$ depends on $\exp[in\theta] + \exp[in(\pi-\theta)]$, so that only $\cos 2n\theta$ and $\sin(2n+1)\theta$ arise; these functions are polynomials in $\sin\theta$. Similarly for bounded obstacles,

$$P_n^m(\cos\theta) + P_n^m(-\cos\theta) = [1 + (-1)^{n+m}]P_n^m(\cos\theta)$$

selects P_{2n}^{2m} *and* P_{2n+1}^{2m+1}, and these too are polynomials in $\sin\theta$. The exponent $k(\hat{\mathbf{r}}_c - \hat{\mathbf{r}}_0)\cdot\mathbf{R} = k(\hat{\mathbf{r}}_c' - \hat{\mathbf{r}}_0)\cdot\mathbf{R} \equiv \mathbf{K}_c\cdot\mathbf{R}$ also depends only on $\sin\theta_c$ and φ_c, so that $e^{i\mathbf{K}_c\cdot\mathbf{R}}\mathscr{F}(r_c)$ is symmetrical to reflection in $z = 0$.

We introduce the Fourier transform of the pair correlation function

$$P[K_c] = \rho\int d\,\mathbf{R} p(\mathbf{R}) e^{i\mathbf{K}_c\cdot\mathbf{R}},$$
$$\mathbf{K}_c = (\mathbf{k}_c - \mathbf{k}_0)\cdot(\hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}}) = k(\alpha_c - \alpha_0)\hat{\mathbf{x}} + k(\beta_c - \beta_0)\hat{\mathbf{y}}, \tag{15}$$

a real function of K_c for elliptical symmetry, and briefer notation $G(\hat{\mathbf{r}},\hat{\mathbf{r}}_0) = G_{r0}, G(\hat{\mathbf{r}}_c,\hat{\mathbf{r}}_0) = G_{c0}$, etc. Thus the mean form of (14) may be written essentially as (5:36).

$$G_{r0} = g_{r0} + \mathrm{S}(g_{rc}G_{c0} + g_{rc'}G_{c'0})$$
$$\mathrm{S} = \lim_{\epsilon\to 0}\frac{1}{2}\int_c e^{-k|\epsilon\gamma_c|}P[K_c],$$
$$\mathrm{S} = \mathrm{S}_r + \mathrm{S}_e,\quad \mathrm{S}_r = \mathrm{Re}\,\mathrm{S} = \tfrac{1}{2}\mathscr{M}_r P[K_c]. \tag{16}$$

As before for the periodic cases, S_r corresponds to real θ_c (the propagating range), and S_e to $|\theta_c| > \pi/2$ (the evanescent range). The operator $\mathscr{M}_r$ representing the mean over the right half-space ($|\theta| < \pi/2$) can be replaced by $\mathscr{M}$ of (5) because of the symmetry of the operand. The operator S_e is imaginary; using $\theta_c = \pm(\pi/2 - i\tau)$ yields real integrals times i.

Equivalently, in terms of the Fourier transform of the total correlation function $p(\mathbf{R}) - 1$,

$$\mathscr{P}[K] = \rho\int[p(\mathbf{R}) - 1]e^{i\mathbf{K}\cdot\mathbf{R}}d\mathbf{R}, \tag{17}$$

which differs from $P[K]$ by the δ-function contributions at $\hat{\mathbf{r}}_0, \hat{\mathbf{r}}_0'$, we make specular terms of S explicit. Thus

$$G_{r0} = g_{r0} + C[g_{r0}G_{00} + g_{r0'}G_{0'0}]$$
$$+ \mathrm{S}^{(0)}(g_{rc}G_{c0} + g_{rc'}G_{c'0}), \tag{18}$$

where $\mathrm{S}^{(0)}$ differs from S of (16) in containing $\mathscr{P}[K]$ instead of $P[K]$. If the effects of $\mathrm{S}^{(0)}$ are negligible, then (18) reduces to the form analyzed before for sparse uncorrelated distributions. The earlier construction consisted essentially of taking the multiple-scattered excitation of one obstacle in the array as the mean of the average transmitted and reflected fields $\frac{1}{2}[\langle\Psi_>\rangle + \langle\Psi_<\rangle]$ of (12).

In Ref. 5 for correlated cylinders, we emphasized reduction procedures of (16) that led to earlier results for the periodic and uncorrelated cases, but now we exclude periodic distributions (except for the low-frequency case of one propagating mode) and emphasize the effects of correlations. To motivate the reduction for G, we first consider the averages of the quadratic function of the field.

III. AVERAGE ENERGY FUNCTIONS

For a fixed configuration, the normalized average energy density is given by $|\Psi|^2$ and the corresponding flux density by $\mathrm{Re}(\Psi^*\nabla\Psi/ik)$ with Ψ as in (7). The ensemble average of $|\Psi|^2$ equals

$$\langle|\Psi|^2\rangle = |\langle\Psi\rangle|^2 + V,$$
$$V = \rho\int\langle|U_s|^2\rangle_s\, d\mathbf{R}_s + \rho^2\int\int[\langle U_s U_t^*\rangle_{st}p(\mathbf{R}_{ts})$$
$$- \langle U_s\rangle_s\langle U_t^*\rangle_t]e^{-i\mathbf{k}\cdot\mathbf{R}_{ts}}d\mathbf{R}_s\, d\mathbf{R}_t, \tag{19}$$

with $|\langle\Psi\rangle|^2 = |\phi + \langle\mathscr{U}\rangle|^2$ in terms of (12) as the coherent intensity, and the variance $V = \langle|\Psi - \langle\Psi\rangle|^2\rangle$ as the incoherent or fluctuation intensity. Substituting the spectral form of U_s, and integrating over $\mathbf{R}_s$ to obtain δ-functions which eliminate one of the spectral operators, we obtain for $z = |z|$,

$$V_> = \int_c 2C_c e^{-2\,\mathrm{Im}\,\gamma_c z}\Big(\langle|G_s(\hat{\mathbf{r}}_c,\hat{\mathbf{k}})|^2\rangle_s$$
$$+ \rho\int d\,\mathbf{R}\, e^{i\mathbf{K}_c\cdot\mathbf{R}}[p(\mathbf{R})\langle G_s G_t^*\rangle_{st} - \langle G_s\rangle\langle G_t^*\rangle_t]\Big), \tag{20}$$

where $C_c = \ \ _0/\gamma_c^*$. For $z\sim\infty$, we have $V_> \sim 2C\gamma_0\mathscr{M}_r\sec\theta\{\ \ \}$. Replacing $\langle G_s G_t^*\rangle_{st}$ by $\langle G_s\rangle_s\langle G_t^*\rangle_t$ with $\langle G_s\rangle_s = G$,

$$V_> \sim 2C\gamma_0\mathscr{M}_r\mathscr{W}[K]|G(\hat{\mathbf{r}},\hat{\mathbf{k}})|^2\sec\theta \equiv \rho\int_> Q_{r0}\sec\theta;$$
$$Q_{r0} = c_1\mathscr{W}|G_{r0}|^2;\ c_1 = \frac{2}{\pi k},\frac{1}{k^2}; \tag{21}$$
$$\mathscr{W}[K] = 1 + \mathscr{P}[K] = 1 + \rho\int(p-1)e^{i\mathbf{K}\cdot\mathbf{R}}d\,\mathbf{R},$$

where $\mathscr{W}[K]$ is the structure factor, and ρQ_{r0} is the multiple scattered differential cross section per unit area. In (21), we integrate $\int_> = \int_> d\theta, d\Omega$ over the forward half-space $|\theta| < \pi/2$; for the back space, we replace $\hat{\mathbf{r}}$ by $\hat{\mathbf{r}}'$, or write $\int_<$ to integrate over the appropriate values of θ.

For the corresponding average flux, we write

$$\mathbf{J} = \mathrm{Re}\langle \Psi^* \nabla \Psi / ik \rangle = \mathbf{J}^c + \mathbf{I}, \quad \mathbf{J}^c = \mathrm{Re}\langle \Psi^* \rangle \nabla \langle \Psi \rangle / ik,$$

$$\mathbf{I}_{\gtrless} \sim \rho \int_{\gtrless} Q_{r0} \hat{\mathbf{r}} \sec\theta; \tag{22}$$

$$\mathbf{J}^c_> = |T|^2 \hat{\mathbf{k}}, \quad \mathbf{J}^c_< = \hat{\mathbf{k}} + \mathrm{Re}(e^{-i2k\gamma_0 z} R)(\hat{\mathbf{k}} + \hat{\mathbf{k}}') + |R|^2 \hat{\mathbf{k}}'.$$

The components normal to the distribution satisfy

$$\hat{\mathbf{z}}\cdot\mathbf{J}^c_> = |T|^2 \gamma_0, \quad \hat{\mathbf{z}}\cdot\mathbf{J}^c_< = (1 + |R|^2)\gamma_0,$$

$$\hat{\mathbf{z}}\cdot(\mathbf{I}_> + \mathbf{I}_<) = \rho\left[\int_> + \int_<\right] Q_{r0} = \rho\sigma_0 \mathscr{M} \mathscr{W} |G_{r0}|^2 = \rho \mathscr{S}_s,$$

with $\rho\mathscr{S}_s$ as the average incoherent scattering cross section per unit area. The corresponding average absorption cross section per unit area is the average inward flux $-\hat{\mathbf{z}}\cdot(\mathbf{J}_> + \mathbf{J}_<) = \rho\mathscr{S}_a$, and conservation of energy is shown by[1,5]

$$1 = |R|^2 + |T|^2 + \rho\mathscr{S}\sec\theta_0, \quad \mathscr{S} = \mathscr{S}_a + \mathscr{S}_s;$$

$$R = 2CG_{0'0}, \quad T = 1 + 2CG_{00}. \tag{23}$$

This states that the incident flux density equals the flux coherently reflected and transmitted, and incoherently scattered and absorbed by the area of distribution irradiated by unit area of incident wave.

To obtain $\mathscr{S}_a$ as a well-defined surface integral of a quadratic function of Ψ, as well as additional relations for G, we write the energy absorbed by a fixed set of scatterers within a closed surface A as

$$-\int \mathrm{Re}\left(\frac{\Psi^* \nabla \Psi}{ik}\right)\cdot d\mathbf{A} = -\frac{1}{2}\sigma_0 \{\Psi^*, \Psi\}_A.$$

Using Green's theorem and $(\nabla^2 + k^2)\Psi = 0$ outside all obstacles we obtain $\{\Psi^*, \Psi\} = \Sigma_t \{\Psi^*, \Psi\}_t$ in order to replace A by the set of scatterers' surfaces (or any set of surfaces that isolates each scatterer from the others). Working initially with $\Psi = \phi + \mathscr{U}$, and then decomposing $\mathscr{U}$, we use Green's theorem and $(\nabla^2 + k^2)U_s = 0$ outside scatterer s to obtain

$$\begin{aligned}\{\Psi^*, \Psi\}_t &= 2\,\mathrm{Re}\{\phi^*, \mathscr{U}\}_t + \{\mathscr{U}^*, \mathscr{U}\}_t \\ &= 2\,\mathrm{Re}\{\phi^*, U_t\}_t + \{U_t^*, U_t\}_t \\ &\quad + 2\,\mathrm{Re}\{U_t^*, \Sigma' U_s e^{-i\mathbf{k}\cdot\mathbf{R}_{st}}\}_t.\end{aligned}$$

Multiplying through by $-\frac{1}{2}\sigma_0$, we relate the absorption cross section of obstacle t to the energy derived from ϕ by interference with U_t, and to the scattering cross section $\frac{1}{2}\sigma_0\{\mathscr{U}^*, \mathscr{U}\}_t$ of obstacle t. The decomposition of $\{\mathscr{U}^*, \mathscr{U}\}_t$ in terms of U_t and U_s, corresponds to an intrinsic component $\{U_t^*, U_t\}_t$ and to a set of two-particle terms. Proceeding essentially as for (8), we obtain

$$\{\Psi^*, \Psi\}_t = 2\,\mathrm{Re}\,G_t(\hat{\mathbf{k}}, \hat{\mathbf{k}}) + 2\mathscr{M}|G_t(\hat{\mathbf{r}}, \hat{\mathbf{k}})|^2 + \mathscr{I},$$

$$\mathscr{I} = 2\,\mathrm{Re}\,\Sigma' e^{-i\mathbf{k}\cdot\mathbf{R}_{st}} \int_c e^{i\mathbf{k}_c\cdot\mathbf{R}_{st}} G_s(\hat{\mathbf{r}}_c, \hat{\mathbf{k}}) G_t^*(\hat{\mathbf{r}}_c^*, \hat{\mathbf{k}}), \tag{24}$$

where the term in $\mathscr{M}$ follows from the asymptotic form $U_s \sim hG_s$ or from the $\int_c$ form in (7). For lossless particles $\{\Psi^*, \Psi\}_t = 0$, Eq. (24) is a special case of (7:A17).

We average (24) with scatterer t fixed for a planar distribution $\mathbf{R}(x,y)$ by proceeding as for (13). Thus, initially we write

$$\begin{aligned}-\mathrm{Re}\langle G_t(\hat{\mathbf{k}}, \hat{\mathbf{k}})\rangle_t &= \mathscr{S}_a/\sigma_0 + \mathscr{M}\langle |G_t(\hat{\mathbf{r}}, \hat{\mathbf{k}})|^2\rangle_t + \mathscr{I}'; \\ \mathscr{I}' &= \mathrm{Re}\,\mathrm{S}[\mathscr{G}_{st}(\hat{\mathbf{r}}_c) + \mathscr{G}_{st}(\hat{\mathbf{r}}_c')], \\ \mathscr{G}_{st}(\hat{\mathbf{r}}_c) &= \langle G_s(\hat{\mathbf{r}}_c, \hat{\mathbf{k}}) G_t^*(\hat{\mathbf{r}}_c^*, \hat{\mathbf{k}})\rangle_{st},\end{aligned} \tag{25}$$

where

$$\mathscr{S}_a = -\tfrac{1}{2}\sigma_0 \langle \{\Psi^*, \Psi\}_t\rangle_t = -\tfrac{1}{2}\sigma_0\{\langle \Psi^*, \Psi\rangle_t\}_t$$

is the average absorption cross section of obstacle t, and where S (in which the implicit $P[K]$ is real) operates on a function of the real variables $\sin\theta_c$ and φ_c. Replacing $\langle G_s G_t^*\rangle_{st}$ by $|G|^2$, etc., and using briefer notation we have

$$\begin{aligned}\mathscr{I}' &= \mathrm{Re}\,\mathrm{S}[|G_{c0}|^2 + |G_{c'0}|^2] \\ &= \mathrm{S}_r[|G_{r0}|^2 + |G_{r'0}|^2] = \mathscr{M} P[K] |G_{r0}|^2,\end{aligned}$$

and make the specular contributions explicit by

$$\mathscr{I}' = \mathscr{M}\mathscr{P}[K]|G_{r0}|^2 + C[|G_{00}|^2 + |G_{0'0}|^2].$$

Thus introducing $\mathscr{I}'$ into (25) and combining the $\mathscr{M}$ terms by using $\mathscr{W} = 1 + \mathscr{P}$, we obtain

$$\begin{aligned}-\mathrm{Re}\,G_{00} &= C[|G_{00}|^2 + |G_{0'0}|^2] + (\mathscr{S}_s + \mathscr{S}_a)/\sigma_0, \\ \mathscr{S}_s &= \sigma_0 \mathscr{M}\mathscr{W}[K]|G_{r0}|^2,\end{aligned} \tag{26}$$

which also follows from (23) on writing R and T in terms of G and using $\rho\sec\theta_0/4C = 1/\sigma_0$.

Similarly, corresponding essentially to (7:A16), we have

$$\begin{aligned}\{\Psi^*(\hat{\mathbf{k}}_1), \Psi(\hat{\mathbf{k}}_2)\}_t &= G_t^*(\hat{\mathbf{k}}_2, \hat{\mathbf{k}}_1) + G_t(\hat{\mathbf{k}}_1, \hat{\mathbf{k}}_2) \\ &\quad + 2\mathscr{M} G_t(\hat{\mathbf{r}}, \hat{\mathbf{k}}_2) G_t^*(\hat{\mathbf{r}}, \hat{\mathbf{k}}_1) + \mathscr{I}_{21} + \mathscr{I}_{12}^*,\end{aligned}$$

$$\mathscr{I}_{21} = \Sigma_s' e^{-i\mathbf{k}_2\cdot\mathbf{R}_{st}} \int_c e^{i\mathbf{k}_c\cdot\mathbf{R}_{st}} G_s(\hat{\mathbf{r}}_c, \hat{\mathbf{k}}_2) G_t^*(\hat{\mathbf{r}}_c^*, \hat{\mathbf{k}}_1). \tag{27}$$

For planar distributions $\mathbf{R}(\hat{\mathbf{x}}, \hat{\mathbf{y}})$, and $\hat{\mathbf{k}}_1, \hat{\mathbf{k}}_2$ each corresponding to either $\hat{\mathbf{k}}$ or $\hat{\mathbf{k}}'$, we have $\mathbf{k}_1\cdot\mathbf{R} = \mathbf{k}_2\cdot\mathbf{R}$, and can proceed essentially as for (24) in terms of the same $P[K]$. Thus

$$\begin{aligned}&-\langle G_t(\hat{\mathbf{k}}_1, \hat{\mathbf{k}}_2)\rangle_t - \langle G_t^*(\hat{\mathbf{k}}_2, \hat{\mathbf{k}}_1)\rangle_t \\ &\quad = 2\mathscr{S}_a^{21}/\sigma_0 + 2\mathscr{M}\langle G_t(\hat{\mathbf{r}}, \hat{\mathbf{k}}_2) G_t^*(\hat{\mathbf{r}}, \hat{\mathbf{k}}_1)\rangle_t + \mathscr{I}_{21}' + \mathscr{I}_{12}'^*; \\ &\mathscr{I}_{21}' = \mathrm{S}[\mathscr{G}_{st}^{21}(\hat{\mathbf{r}}_c) + \mathscr{G}_{st}^{21}(\hat{\mathbf{r}}_c')] \equiv \mathrm{S}\mathscr{F}_{st}^{21}, \\ &\mathscr{G}_{st}^{21}(\hat{\mathbf{r}}_c) = \langle G_s(\hat{\mathbf{r}}_c \hat{\mathbf{k}}_2) G_t^*(\hat{\mathbf{r}}_c, \hat{\mathbf{k}}_1)\rangle_{st}: \quad \mathscr{I}_{12}' = (\mathscr{F}_{ts}^{21})^*,\end{aligned} \tag{28}$$

where

$$-2\mathscr{S}_a^{21} = \sigma_0\{\langle \Psi^*(\hat{\mathbf{k}}_1), \Psi(\hat{\mathbf{k}}_2)\rangle_t\}_t.$$

Dropping the dependence on s, t as before, we write $\mathscr{G}_c = G_{c2} G_{c1}^*$ and $\mathscr{F} = \mathscr{G}_c + \mathscr{G}_{c'}$, to obtain

$$\begin{aligned}\mathscr{I}_{21}' + \mathscr{I}_{12}'^* &= \mathrm{S}\mathscr{F} + [\mathrm{S}\mathscr{F}^*]^* = (\mathrm{S} + \mathrm{S}^*)\mathscr{F} = 2\mathrm{S}_r\mathscr{F} \\ &= 2\mathscr{M} P[K] G_{r2} G_{r1}^* \\ &= 2\mathscr{M}\mathscr{P}[K] G_{r2} G_{r1}^* \\ &\quad + 2C[G_{02}G_{01}^* + G_{0'2}G_{0'1}^*],\end{aligned}$$

which reduces (28) to

$$- G_{12} - G^*_{21} = 2C\,[G_{02}G^*_{01} + G_{0'2}G^*_{0'1}] + 2\mathscr{M}\mathscr{W}[K]G_{r2}G^*_{r1} + 2\mathscr{S}^{21}_a/\sigma_0. \tag{29}$$

Equation (29) represents four relations, with (26) as the case $\hat{\mathbf{r}}_1 = \hat{\mathbf{r}}_2 = \hat{\mathbf{r}}_0 = \hat{\mathbf{k}}$.

Similarly from $\{\langle\Psi(\hat{\mathbf{r}}_1),\Psi(\hat{\mathbf{r}}_2)\rangle,\}_t = 0$, we obtain the reciprocity relation

$$G(\hat{\mathbf{r}}_1,\hat{\mathbf{r}}_2) = G(-\hat{\mathbf{r}}_2 - \hat{\mathbf{r}}_1), \quad G_{12} = G_{-2-1},$$

for $\hat{\mathbf{r}}_1$ and $\hat{\mathbf{r}}_2$ each corresponding to either $\hat{\mathbf{r}}_0$ or $\hat{\mathbf{r}}'_0$.

From (29), in terms of

$$T = 1 + 2CG_{00}, \quad T' = 1 + 2CG_{0'0'},$$
$$R = 2CG_{0'0}, \quad R' = 2CG_{00'};$$
$$S_{12} = (\mathscr{S}^{12}_a = \mathscr{S}^{12}_s)\rho \sec\theta_0;$$
$$S = S_{00}, \quad S' = S_{0'0'}, \quad S'' = S_{0'0} = S^*_{00'},$$

we obtain four equalities

$$1 = |R|^2 + |T|^2 + S = |R'|^2 + |T'|^2 + S',$$
$$0 = (R'T^* + R^*T' + S'') = (\ )^*. \tag{30}$$

The first two correspond to conservation of energy for incident ϕ or ϕ', and the second two provide additional constraints. For lossless particles and negligible fluctuation effects (or for the one-propagating mode periodic cases), we drop the S terms to obtain the earlier results discussed in detail for periodic[12] structures; for such cases, $|T| = |T'|$ and $|R| = |R'|$, and the sum of the transmitted phases differ by 180° from the sum of the reflected phases. For a symmetrical monolayer ($R = R'$, $T = T'$), (30) reduces to two relations

$$1 = |R|^2 + |T|^2 + S, \quad 2\,\mathrm{Re}\,R^*T + S'' = 0;$$
$$1 = |T \pm R|^2 + (S \pm S''), \tag{31}$$

with S and S'' real. If S is negligible, then $\mathrm{Re}\,R^*T = 0$, and R and T are 90° out of phase; for such cases $|T \pm R| = 1$.

IV. SCATTERING AMPLITUDE REDUCTIONS

The procedure we followed to obtain the generalized energy relation (29) corresponding to G of (18), may be applied to a modified scattering problem shorn of specular losses specified by

$$\not{g}_{r0} = g_{r0} + \mathrm{S}^{(0)}[g_{rc}\not{g}_{c0} + g_{rc'}\not{g}_{c'0}]. \tag{32}$$

For lossless scatterers, the procedure yields

$$\not{g}_{12} + \not{g}^*_{21} = -2\mathscr{M}\mathscr{W}[K]\not{g}_{r2}\not{g}^*_{r1},$$
$$\mathscr{W}[K] = 1 + \mathscr{P}[K] = 1 + \rho\int[p(\mathbf{R}) - 1]e^{i\mathbf{K}\cdot\mathbf{R}}\,d\mathbf{R}, \tag{33}$$

where $\mathbf{K}\cdot\mathbf{R} = k(\hat{\mathbf{r}} - \hat{\mathbf{r}}_1)\cdot\mathbf{R} = k(\hat{\mathbf{r}} - \hat{\mathbf{r}}_2)\cdot\mathbf{R}$, and $\hat{\mathbf{r}}_1$ or $\hat{\mathbf{r}}_2$ each represents either $\hat{\mathbf{k}}$ or $\hat{\mathbf{k}}'$. In terms of $\not{g}$, we reduce (18) to

$$G_{r0} = \not{g}_{r0}(1 + CG_{00}) + \not{g}_{r0'}CG_{0'0}, \tag{34}$$

the form considered earlier[1] in terms of g. The interpretation of $G[\not{g}]$ is the same as before in that G corresponds to $\not{g}$ excited by the mean of the averge transmitted and reflected fields of (12), $\frac{1}{2}[\langle\Psi_>\rangle + \langle\Psi_<\rangle]$, evaluated at $z = 0$. The following indicates the physics implicit in $\not{g}$ to provide a guide for developing practical forms for data inversion.

The analog of (33) for a lossless isolated obstacle is

$$g_{12} + g^*_{21} = -2\mathscr{M}g_{r2}g^*_{r1} = -2\mathscr{M}g_{1r}g^*_{2r}, \tag{35}$$

where $g_{12} = g(\hat{\mathbf{r}}_1,\hat{\mathbf{r}}_2)$ with $\hat{\mathbf{r}}_1$ and $\hat{\mathbf{r}}_2$ arbitrary; the second equality follows from reciprocity and the fact that $\mathscr{M}f(\hat{\mathbf{r}}) = \mathscr{M}f(-\hat{\mathbf{r}})$. The special value $\mathrm{Re}\,g_{00} = -\mathscr{M}|g_{r0}|^2$ corresponds to (5) for $\sigma_a = 0$. To develop approximations of g as well as of the multiple scattered amplitude for periodic distributions,[3,4] we expressed g before in terms of the amplitude g' for the radiationless problem

$$g_{12} = g'_{21} + \mathscr{M}g'_{1r}g_{r2} = g'_{12} + \mathscr{M}g_{1r}g'_{r2} \tag{36}$$

such that for the lossless case, $g'_{12} + g'^*_{21} = 0$. The leading term of (36) includes phase and absorption effects (that dominate for various special cases) but no radiative losses.

Decomposing the operator $\mathrm{S}^{(0)}$ in (32) as

$$\tfrac{1}{2}\mathscr{M}_r\mathscr{P} + \mathrm{S}_e = \tfrac{1}{2}\mathscr{M}_r\mathscr{W} - \tfrac{1}{2}\mathscr{M}_r + \mathrm{S}_e,$$

and representing the operand by $[g\not{g}]$, we use $\mathscr{M}f(\hat{\mathbf{r}}) = \frac{1}{2}\mathscr{M}_r[f(\hat{\mathbf{r}}) + f(\hat{\mathbf{r}}')]$ for (36) to construct

$$\not{g}_{r0} = g_{r0} + [\tfrac{1}{2}\mathscr{M}_r\mathscr{P} + \mathrm{S}_e][g\not{g}]$$
$$= g'_{r0} + [\tfrac{1}{2}\mathscr{M}_r\mathscr{W} + \mathrm{S}_e][g'\not{g}].$$

Suppressing S_e by introducing the radiationless distribution amplitude

$$g''_{r0} = g'_{r0} + \mathrm{S}_e[g'g''], \tag{37}$$

such that for the lossless case $g''_{12} + g''^*_{21} = 0$ with $\hat{\mathbf{r}}_1$ or $\hat{\mathbf{r}}_2$ each either $\hat{\mathbf{k}}$ or $\hat{\mathbf{k}}'$, we obtain

$$\not{g}_{12} = g''_{12} + \mathscr{M}\mathscr{W}[K][g''_{1r}\not{g}_{r2}] \tag{38}$$

as the analog of the isolated obstacle result (36).

For large kb, the operation over S_e is negligible for the present problem (which excludes complex θ_0, as well as multimode periodic cases). We may also neglect S_e for small kb, provided that the scatterer's width is small compared to minimum separation (a/b small); if not, then as shown subsequently, S_e may introduce k-independent factors arising from multipole coupling. Neglecting S_e corresponds to $\not{g}_{12} \approx g'_{12} + \mathscr{M}\mathscr{W}g'_{1r}\not{g}_{r2}$ which differs from g_{12} in that the radiation integral is modulated by the structure factor $\mathscr{W} = 1 + \mathscr{P}$; the first term of $\mathscr{W}$ corresponds to radiation over the continuum as for the isolated case, the $\mathscr{P}$ term corresponds to the effects of correlations, and their sum represents fluctuation scattering. The similarities in the forms for $\not{g}$ and g indicate how we may carry over existing results by inspection of g. In particular, for various practical problems for small scatterers, we may construct approximations for $\not{g}$ by using phase and absorption terms of g and by modifying the radiation term of g by incorporation $\mathscr{W}$.

We reduce (34) by eliminating G from the right-hand side, and write

$$G_{r1} = [\not{g}_{r1}(1 - C\not{g}_{22}) + C\not{g}_{r2}\not{g}_{21}]/D,$$
$$D = (1 - C\not{g}_{11})(1 - C\not{g}_{22}) - C^2\not{g}_{12}\not{g}_{21}$$
$$\equiv 1 - C(\not{g}_{11} + \not{g}_{22}) + C^2||\not{g}||;$$
$$G_{11} = (\not{g}_{11} - C||\not{g}||)/D, \quad G_{21} = \not{g}_{21}/D, \tag{39}$$

where 1,2 correspond to 0,0′ for incident ϕ, and to 0′,0 for incident ϕ'. If we construct $\mathscr{M}\mathscr{W}G_{r2}G^*_{r1}$ (for r_1 and r_2 different or equal) and apply (33) to eliminate integrals of products of $\wp$'s, we rewrite the results collectively as

$$-G_{12}-G^*_{21}$$
$$=2C\,[G_{02}G^*_{01}+G_{0'2}G^*_{0'1}]+2\mathscr{M}\mathscr{W}G_{r2}G^*_{r1}, \quad (40)$$

i.e., as (29) for $\mathscr{S}_a=0$. If $\hat{\mathbf{r}}_1=\hat{\mathbf{r}}_2$, we obtain the lossless version of (26)

$$-\operatorname{Re}G_{11}=C(|G_{01}|^2+|G_{0'1}|^2)+\mathscr{M}\mathscr{W}|G_{r1}|^2, \quad (41)$$

where 1 equals 0 or 0′. In the isolated obstacle result $-\operatorname{Re}g_{11}=M|\wp_{r1}|^2$, the term $-\operatorname{Re}g_{11}$ corresponds to the energy lost from the incident wave via interference with the scattered wave, and $\mathscr{M}|g_{r1}|^2$ shows it is balanced by radiation over all directions. In (41), the interference loss is balanced by specular scattering in the forward and reflected directions, and by fluctuation scattering over all directions. Equation (41) is the same form as (5:62) for the two-dimensional case; using the Zernike–Prins[6] distribution function, we showed before[5] that it reduced to the required forms for the uncorrelated and periodic limits.

From (39), we construct the transmission and reflection coefficients,

$$T_{11}=(1+C\wp_{11}-C\wp_{22}+C^2||\wp||)/D, \quad R_{21}=2C\wp_{21}/D,$$
$$D=1-C(\wp_{11}+\wp_{22})+C^2||\wp||. \quad (42)$$

The corresponding differential scattering cross section per unit area, $\rho Q_{r1}\approx\rho c_1\mathscr{W}[K]|G_{r1}|^2$ is determined by

$$|D|^2|G_{r1}|^2=|\wp_{r1}|^2|1-C\wp_{22}|^2+|\wp_{r2}|^2|C\wp_{21}|^2$$
$$+2\operatorname{Re}\wp^*_{r1}\wp_{r2}(1-C\wp^*_{22})C\wp_{21}. \quad (43)$$

Using these and (30) we write the transmitted flux as

$$|T_{11}|^2=1-|R_{21}|^2-\rho\mathscr{S}_{11}\sec\theta_0,$$
$$|T_{11}|^2=1+4C\operatorname{Re}[(\wp_{11}-C||\wp||)(1-C\wp^*_{22})]/|D|^2,$$
$$\mathscr{S}_{11}|D|^2=\bar{\sigma}_{11}|1-C\wp_{22}|^2+\bar{\sigma}_{22}|C\wp_{21}|^2$$
$$+2\operatorname{Re}\bar{\sigma}_{21}(1-C\wp^*_{22})C\wp_{21};$$
$$\bar{\sigma}_{21}\equiv-(\wp^*_{21}+\wp_{12})\sigma_0/2, \quad \bar{\sigma}=\bar{\sigma}_s+\bar{\sigma}_a. \quad (44)$$

For grazing incidence, $\cos\theta_0\to0$ and $C\to\infty$, so that $G\to0$ and $T\to1$. Except near grazing, for small $|C\wp|$, we have

$$T\approx1+2C\wp_{11}+2C^2(\wp^2_{11}+\wp_{21}\wp_{12}), \quad R\approx2\wp_{21}C,$$
$$|T|^2\approx1-|2C\wp_{21}|^2+4C\operatorname{Re}\wp_{11}\approx1-|R|^2-\rho\bar{\sigma}_{11}\sec\theta_0,$$
$$Q_{r1}\approx\bar{q}_{r1}=c_1\mathscr{W}[K]|\wp_{r1}|^2, \quad (45)$$

where we neglected

$$\operatorname{Re}(\wp^2_{11}+|\wp_{11}|^2+\wp_{12}\wp_{21}+|\wp_{21}|^2)$$
$$=2\operatorname{Re}(\wp_{11}\bar{\sigma}_{11}+\wp_{21}\bar{\sigma}_{21})=\mathscr{O}(\sigma^2).$$

For small $\operatorname{Im}g'\approx g\gg\operatorname{Re}g$, or small $\mathscr{W}$ and small absorption, we iterate (37) for $g''\approx g'$ and neglect absorption in the quadratic terms, to obtain

$$\wp_{11}\approx g'_{11}+\mathscr{M}\mathscr{W}g'_{1r}g'_{r1}=g'_{11}-\mathscr{M}\mathscr{W}|g'_{r1}|^2$$
$$\approx i\operatorname{Im}g_{11}-(\sigma_a+\bar{\sigma}_s)/\sigma_0,$$
$$\bar{\sigma}_s=\mathscr{M}\mathscr{W}[K]|g_{r0}|^2,$$
$$|T|^2\approx1-|2C\operatorname{Im}g_{21}|^2-\rho(\sigma_a+\bar{\sigma}_s)\sec\theta_0, \quad (46)$$

such that for $\mathscr{W}\approx1$,$\bar{\sigma}_s$ approximates σ_s. The corresponding differential scattering cross section $Q_{r0}\approx\mathscr{W}[K]q_{r0}$, with $q_{r0}=c_1|g_{r0}|^2$ as the value for an isolated obstacle, may be used to isolate $\mathscr{W}[K]$ from measurements essentially as in x-ray diffraction by liquids. For the direct problem we use the Zernike–Prins[6] p and $\mathscr{W}[K]$ for the planar distribution of parallel cylinders; for bounded obstacles, we have explicit results for restricted ranges and numerical results for p based on the Percus–Yevick or other integral equation approximations.[9]

For kb small (small-spaced particles),

$$\mathscr{W}[K]=\mathscr{W}[0]+O(k^2),$$
$$\mathscr{W}[0]=\mathscr{W}=1+\rho\int[p(R)-1]dR, \quad (47)$$

where $\mathscr{W}$ is proportional to the variance (fluctuation) of the number of particles in a central region. We obtained $\mathscr{W}$ in closed form [10,11] by using statistical mechanics theorems and the scaled particle approximation for the equations of state[13] for fluids of impenetrable particles. For planar distributions of correlated parallel strips ($\mathscr{W}_1$) and disks ($\mathscr{W}_2$), and for lattice gas statistics ($\mathscr{W}_0$), we showed

$$\mathscr{W}_0=1-w, \quad \mathscr{W}_1=(1-w)^2, \quad \mathscr{W}_2=\frac{(1-w)^3}{1+w}, \quad (48)$$

where $w=\rho v$ is the fraction of the xy plane covered by statistical mechanics particles of width or area v. The upper bound on w is unity for $\mathscr{W}_1$ and $\mathscr{W}_0$; for $\mathscr{W}_2$ we use $w\lesssim0.84$, as measured for circular disks. The associated function $S=w\mathscr{W}$ has a maximum $S_\cap$ at $w=w_\cap$ determined by $\partial_w S=0$, and so does the corresponding differential scattering cross section per unit area ($\propto S_\cap$). If the particles are nonabsorbing, the sum of the coherent fluxes $|T^2+|R|^2=1-\rho\mathscr{W}\sigma_s\sec\theta_0$ has a minimum at $w_\cap$; if they are absorbing, and σ_a/σ_s is small then the minimum is shifted to

$$w_\wedge=w_\cap+\sigma_a/\sigma_s|\partial^2_w S_\cap|>w_\cap;$$

if σ_a/σ_s is not small then there is in general no minimum with variation of w, and σ_a dominates for all w. See analogous discussion of random volume distributions for details.

For some purposes we may compare monolayer results with equivalents for single-path transmission through a slab region volume distribution. Multiplying numerator and denominator of $2C\wp_{11}=\rho\sigma_0\wp_{11}/2\cos\theta_0$ by the layer thickness d_0 (the separation of the tangent planes), we define an equivalent complex index of refraction η by

$$2C\wp_{11}=\frac{\rho}{d_0}\left(\frac{\sigma_0\wp_{11}}{2}\right)d_0\sec\theta_0\equiv ik(\eta-1)d\equiv i\Lambda,$$
$$d=d_0\sec\theta_0, \quad (49)$$

where ρ/d_0 is the number of particles per unit volume, and d is the ray path along the direction of incidence. Similarly,

$R = 2C\wp_{21} \equiv iFkd$ defines $-F$ as an equivalent Fresnel interface coefficient.

Introducing (49) in (45) and (46), we exponentiate $i\Lambda$, and work to lowest order with

$$\begin{aligned} &T \approx e^{i\Lambda}(1 + \tfrac{1}{2}|R|^2),\\ &|T|^2 \approx e^{-2\,\mathrm{Im}\,\Lambda}(1 - |R|^2) \approx 1 - 2\,\mathrm{Im}\,\Lambda - |R|^2,\\ &i\Lambda \approx i2C\,\mathrm{Im}\,g_{11} - \rho(\sigma_a + \mathscr{W}\sigma_s)/2\cos\theta_0,\\ &|R|^2 \approx |2C\,\mathrm{Im}\,g_{21}|^2, \end{aligned} \tag{50}$$

where we would use $\sigma_s = \mathscr{M}\mathscr{W}[K]|\mathrm{Im}\,g_{11}|^2$ if the scatterers were not small spaced. To lowest order in k, except for pressure release scatterers, from $g'_{r1} = i\Gamma[\mathscr{F}_{r1} + \mathscr{O}(k^2)]$ with $\Gamma = k\mathscr{V}/\sigma_0 = \mathscr{O}(k^2, k^3)$ and $\mathscr{F}$ independent of k, we have

$$\begin{aligned} &\mathrm{Re}\,\Lambda \approx \rho\mathscr{V}k\,\mathrm{Re}\,\mathscr{F}_{11}/2\gamma_0,\\ &\mathrm{Im}\,\Lambda \approx (\rho\mathscr{V}k/2\gamma_0)[\mathrm{Im}\,\mathscr{F}_{11} + \Gamma\mathscr{W}\mathscr{M}(\mathrm{Re}\,\mathscr{F}_{r1})^2],\\ &|R|^2 \approx |\mathrm{Re}\,\mathscr{F}_{21}\rho\mathscr{V}k/2\gamma_0|^2. \end{aligned} \tag{51}$$

For such scatterers, $|R|^2 = \mathscr{O}(k^2)$ dominates $\sigma_s = \mathscr{O}(k^3, k^4)$ in the scattering losses of a transmitted ray, except for special angles corresponding to small $\mathscr{F}_{21}$.

The limitations of (50) and (45) are indicated by the complete expression T of (42). For detailed computations, we use

$$\begin{aligned} &T = |T|e^{i\Theta_T} = \frac{1+Z}{1-Z}, \quad |T|^2 = 1 + \frac{4\,\mathrm{Re}\,Z}{|1-Z|^2},\\ &\tan\Theta_T = \frac{\mathrm{Im}\,T}{\mathrm{Re}\,T} = \frac{2\,\mathrm{Im}\,Z}{1-|Z|^2},\\ &Z = C\wp_{11} + C^2\wp_{12}\wp_{21}/(1 - C\wp_{22})\\ &\quad = C\wp_{11} + C^2\wp_{12}\wp_{21}(1 + C\wp_{22}) + \mathscr{O}(k^4). \end{aligned} \tag{52}$$

For σ_a at least as small as σ_s, the transmitted phase to $\mathscr{O}(k^3)$ in terms of $\tau_{ij} = C\,\mathrm{Im}\,g''_{ij}$ with $g'' \approx g'$ for small a/b is given by

$$\begin{aligned} &\tan\Theta_T \approx \frac{2(\tau_{11} - \tau_{12}\tau_{21}\tau_{22})}{1-\tau_{11}^2} \approx 2(\tau_{11} + \tau_{11}^3 - \tau_{12}\tau_{21}\tau_{22}),\\ &\Theta_T \approx 2(\tau_{11} - \tau_{12}\tau_{21}\tau_{22} - \tau_{11}^3/3). \end{aligned} \tag{53}$$

This provides a more complete result than in (51). Similarly from the complete form of R of (42), we specify the reflected phase by $\mathrm{Im}\,R/\mathrm{Re}\,R = \tan\Theta_R$ such that

$$\begin{aligned} \tan\Theta_R &\approx -\frac{\tau_{21}(1 - \tau_{11}\tau_{22} + \tau_{12}\tau_{21})}{\tau_{21}(\tau_{22} + \tau_{11}) + C\,\mathrm{Re}\,\wp_{21}}\\ &\approx -\frac{1 - \tau_{11}\tau_{22} + \tau_{21}\tau_{12}}{\tau_{22} + \tau_{11}},\\ \Theta_R &\approx (\tau_{11} + \tau_{22})(1 - \tau_{12}\tau_{21}) - (\tau_{11}^3 + \tau_{22}^3)/3 + \pi/2, \end{aligned} \tag{54}$$

where we dropped $\mathrm{Re}\,\wp_{21}$ for negligible losses. The set of four phases corresponding to T, R, T', and R' of (30) for negligible S satisfy $\Theta_R + \Theta_{R'} - \Theta_T - \Theta_{T'} = \pi$, where $\Theta_{T'}$ follows from Θ_T of (53) by interchanging 1 and 2, and where $\Theta_{R'} = \Theta_R$ of (54). Thus, to the present degree of approximation, the sum of the reflected phases differs by 180° from the sum of the transmitted phases as discussed earlier for the lossless periodic case (12:15). Similarly for negligible losses we require $|T| = |T'|$ and $|R| = |R'|$; this will be demonstrated in terms of explicit results for $\wp$ we obtain subsequently.

The general forms (39) and (42) can be simplified considerably for normal incidence $\hat{\mathbf{k}} = \hat{\mathbf{z}}$ and arbitrary scatterers, or for scatterers symmetrical to reflection in the plane of centers and arbitrary $\hat{\mathbf{k}}$. For the first case, with $\hat{\mathbf{k}}_1 = -\hat{\mathbf{k}}_2 = \hat{\mathbf{z}}$, the reciprocity theorem (6) gives $g_{11} = g_{22}$, and although g_{21} need not equal g_{12}, we may factor the denominator to obtain

$$\begin{aligned} &T = T_{11} = \Gamma_{22} = \tfrac{1}{2}[\mathscr{R}_+ + \mathscr{R}_-],\\ &\frac{R_{21}}{\kappa} = \kappa R_{12} = R = \tfrac{1}{2}[\mathscr{R}_+ - \mathscr{R}_-], \quad \mathscr{R}_\pm = \frac{1+Z_\pm}{1-Z_\pm};\\ &Z_\pm = C(\wp \pm \wp'), \quad \wp = \wp_{11} = \wp_{22}, \quad \wp' = (\wp_{12}\wp_{21})^{1/2},\\ &\kappa = \left(\frac{\wp_{21}}{\wp_{12}}\right)^{1/2};\\ &G_{r1} = \frac{\wp_{r1} + \kappa\wp_{r2}}{2(1-Z_+)} + \frac{\wp_{r1} - \kappa\wp_{r2}}{2(1-Z_-)}. \end{aligned} \tag{55}$$

This is restricted to normal incidence, but the scatterers' shape and orientation are arbitrary. If the scatterers have inversion symmetry, then $\wp_{21} = \wp_{12} = \wp'$ and $\kappa = 1$ for arbitrary orientation with respect to $z = 0$.

The same simple forms as for $\kappa = 1$ also apply for arbitrary $\hat{\mathbf{k}}$ for scatterers having reflection symmetry in $z = 0$, with no requirement of inversion symmetry. For $\wp_{r1} = \wp_{r1'}$,

$$\begin{aligned} &G_{r0} = \tfrac{1}{2}(F_{r0}^+ + F_{r0}^-), \quad F_{r0}^\pm = f_{r0}^\pm/(1 \pm Z_\pm),\\ &f_{r0}^\pm = \wp_{r0} \pm \wp_{r0'}, \; Z_\pm = Cf_{00}^\pm = C(\wp_{00} \pm \wp_{00'});\\ &T = 1 + \frac{Z_+}{1-Z_+} + \frac{Z_-}{1-Z_-} = \frac{1 - Z_+Z_-}{D},\\ &R = \frac{Z_+}{1-Z_+} - \frac{Z_-}{1-Z_-} = \frac{Z_+ - Z_-}{D},\\ &D = 1 - Z_+ - Z_- + Z_+Z_-. \end{aligned} \tag{56}$$

The functions f^+ and f^- are twice the components of $\wp$ symmetrical and antisymmetrical with respect to reflection of $\hat{\mathbf{k}}$ (or $\hat{\mathbf{r}}$) in the plane $z = 0$. Equivalently, f^+ or f^- are the scattering amplitudes for protuberances on a base plane for which $\partial_z\psi$ or ψ equals zero, a rigid (+) or free (−) base. For ϕ' incident on such structures the corresponding coherent reflected waves are $\pm\phi(1 + Z_\pm)/(1 - Z_\pm)$ with $Z_\pm$ as normalized impedances. We discuss such problems in detail in a sequel.

V. MULTIPOLE EXPANSIONS

For various detailed applications or computations, we reduce $G[g]$ of (16) by expanding the scattering amplitudes G and g in Fourier series. This provides algebraic systems for the corresponding scattering coefficients A and a in terms of distribution integrals $\mathscr{H}$ (continuum analogs of the lattice sums for the periodic cases).[5,6] In Ref. 7, using the general form $G[g]$ we derived the general algebraic systems $A[a]$ for arbitrary configurations, and indicated how the same results

followed from more detailed manipulations starting with expansions of all fields in terms of special functions. We included essentials of special function derivations for the planar periodic cases[3,4] to stress several points, but now we start directly from (16).

A. Cylinders

For cylinders, we use the trigonometric series

$$g_{r0} = \sum_{n=-\infty}^{\infty} a_n e^{in\theta},$$
$$a_n = a_n(\theta_0) = \sum_{m=-\infty}^{\infty} a_{nm} e^{-im\theta_0},$$
$$G_{r0} = \Sigma A_n e^{in\theta},$$

where the angle independent coefficients satisfy $a_{nm} = (-1)^{n+m} a_{-m-n}$ (from the reciprocity theorem). Substituting into (16) we obtain in general

$$A_n = a_n + \sum_{vm} a_{nm} A_v \mathscr{H}_{m-v} \tag{57}$$

and for circular cylinders as in (5:93)

$$A_n = a_n [e^{-in\theta_0} + \Sigma A_m \mathscr{H}_{m-v}], \tag{58}$$

where the a_n are independent of angles. The distribution integrals $\mathscr{H}_n = (-1)^n \mathscr{H}_{-n}$ are given by (5:93) ff,

$$\begin{aligned}\mathscr{H}_n &= \mathrm{S}\{e^{-in\theta_c} + e^{-in(\pi-\theta_c)}\}\\ &= \rho \int_b^\infty dx\, p(x) H_n(kx) [e^{ikx \sin\theta_c}(-1)^n \\ &\quad + e^{-ikx\sin\theta_c}],\end{aligned} \tag{59}$$

with $H_n = H_n^{(1)} = J_n + iN_n$ and corresponding decompositions $\mathscr{H} = \mathscr{J} + i\mathscr{N}$ or $\mathrm{S} = \mathrm{Re}\,\mathrm{S} + \mathrm{S}_c$. The second form in (59), which follows from the first by applying the spectral integral in S to the special cases of

$$\int_c e^{i\mathbf{k}_c\cdot\mathbf{r}} e^{in\theta_c} = i^n H_n(kr) e^{in\theta}$$

at hand, makes the dependence on k and θ_0 explicit. [In Ref. 5 (1953), we derive the second form, as well as (11), by expanding all fields in terms of special functions.]

For even and odd n, we decompose (59) into two sets

$$\begin{aligned}\mathscr{H}_{2n} &= 2\mathrm{S}\cos 2n\theta_c \\ &= 2\int dx\, p(x) H_{2n}(kx) \cos(kx \sin\theta_0),\\ \mathscr{H}_{2n+1} &= -2i\mathrm{S}\sin(2n+1)\theta_c \\ &= -2i \int dx\, p(x) H_{2n+1}(kx) \sin(kx\sin\theta_0).\end{aligned} \tag{60}$$

For normal incidence ($\theta_0 = 0$), the only set that arises is

$$\mathscr{H}_{2n} = 2\int dx\, p(x) H_{2n}(kx). \tag{61}$$

See earlier work[3] for discussion and reduction of the same algebraic systems (57) and (58) in terms of the different $\mathscr{H}$'s (lattice sums) appropriate for the periodic case.

The $\mathscr{H}$'s are determined by the Zernike–Prins pair function[5,6]

$$\begin{aligned}\rho p(x) &= \sum_{s=1}^{[x/b]} \frac{(x-sb)^{s-1}}{D^2(s-1)!} e^{-(x-sb)},\\ D &= \bar{b} - b = (1-w)/\rho,\end{aligned} \tag{62}$$

where $[u]$ is the closest integer to u from below. We may also use the residue-series representation[5]

$$\begin{aligned}\rho p(x) &= \sum_{v=-\infty}^{\infty} \frac{e^{x\gamma_v}}{b + De^{b\gamma_v}}; \quad 1 + D\gamma_v = e^{-\gamma_v b};\\ \gamma_v &= \gamma_{\pm|v|} = -\alpha_{|v|} \pm i\beta_{|v|}; \quad \alpha,\beta > 0\end{aligned}$$

derived by taking[5] the Laplace transform of (62), and then the inverse; see Ref. 5 for simple approximations of the roots near the sparse ($w \approx 0$) and periodic ($w \approx 1$) limits. From geometrical considerations, or directly from (62), we obtain the virial expansion in powers of $w = \rho b$, e.g., to $\mathscr{O}(w^2)$ in terms of $u = x/b$,

$$\begin{aligned}&p = 0 \text{ for } u < 1,\ p = 1 + w(2-u) \text{ for } 1 < u < 2,\\ &p = 1 \text{ for } u > 2.\end{aligned} \tag{63}$$

The Fourier transform of $\rho p(x)$, except near the periodic limit, is given by

$$\begin{aligned}&P[K_c] = \mathscr{P}[K_c] + (2\pi\rho/k)\delta(\sin\theta_c - \sin\theta_0);\\ &\mathscr{P}(\mathscr{K}) = 1 + \mathscr{P}(\mathscr{K})\\ &\quad = 1 + 2w\int_0^\infty (p-1)\cos\mathscr{K}u\, du = (1+\mathscr{D})^{-1},\\ &\frac{\mathscr{D}}{2} = \frac{w\sin\mathscr{K}}{(1-w)\mathscr{K}} + \frac{w^2(1-\cos\mathscr{K})}{(1-w)^2\mathscr{K}^2},\\ &\mathscr{K} = kb(\sin\theta - \sin\theta_0),\end{aligned} \tag{64}$$

where $-\mathscr{D}$ is the transform of the direct correlation function. See Zernike and Prins[6] for the original derivation of $\mathscr{W}$ and for plots of $\mathscr{W}$ and $\rho p(x)$ for several values of $b/\bar{b} = w$. For small kb, or for $\theta \approx \theta_0$, from (64) by expanding $\sin\mathscr{K}$ and $\cos\mathscr{K}$ or from the virial expansion (63),

$$\begin{aligned}&\mathscr{W}(\mathscr{K}) = \mathscr{W} + B\mathscr{K}^2 + \mathscr{O}(\mathscr{K}^4), \quad \mathscr{W} = (1-w)^2,\\ &B = \mathscr{W} w(1 - 3w/4)/3,\end{aligned} \tag{65}$$

with $\mathscr{W} = \mathscr{W}_1$ as in (48). For large $\mathscr{K}(1-w)$, from (64) in terms of $\mathscr{D}$, or from integration by parts and using the scaled particle[13] $p(b)$,

$$\mathscr{W}(\mathscr{K}) \sim 1 - [2w\sin\mathscr{K}/(1-w)\mathscr{K}] \tag{66}$$

and $\mathscr{W}$ tends to unity in an oscillatory fashion as $\mathscr{K}$ increases. We also showed[5] that in the periodic limit $\bar{b} \to b, w \to 1$,

$$\begin{aligned}&P[K] \to -1 + \left(\frac{2\pi}{kb}\right)\sum_{n=-\infty}^{\infty} \delta(\sin\theta_c - \sin\theta_n)\\ &\sin\theta_n = \sin\theta_0 + 2n\pi/kb,\end{aligned}$$

corresponding to the grating of spacing b. The present paper considers only the low-frequency periodic case $kb(1 \pm \sin\theta_0) < 2\pi$, the case of one propagating mode.

We obtain $\mathscr{J}_n$ from the first form in (59) by replacing S by Re S,

$$\mathscr{J}_n = \mathscr{J}_n^0 + \mathscr{J}_n' - \delta_{n0},$$
$$\mathscr{J}_n^0 = C\{e^{-in\theta_0} + e^{-in(\pi-\theta_0)}\},$$
$$\mathscr{J}_n' = \tfrac{1}{2}\mathscr{M}\mathscr{W}[K]\{e^{-in\theta} + e^{-in(\pi-\theta)}\}, \tag{67}$$

where $C = \rho/k\cos\theta_0 = (k\bar{b}\cos\theta_0)^{-1}$, and $\{\ \}$ $= 2\cos 2n\theta, -i2\sin(2n+1)\theta$. The specular contribution $\mathscr{J}_n^0$ is the same as for the one-propagating mode periodic case[3] with b replaced by $\bar{b}$; the term δ_{n0} converts coefficients a of g as in (62) to corresponding coefficients a' of g'. We may also write

$$\mathscr{J}_n' - \delta_{n0} = \tfrac{1}{2}\mathscr{M}\mathscr{P}\{\ \} = \rho\int dx(p-1)J_n[\ ]$$

with $[\ ]$ as in the second form in (59). For small kb, from (64) and (67), or from $\mathscr{J}(J)$ with $J_n(x)\approx(x/2)^n/n!$, we have

$$\mathscr{J}_0' \approx \mathscr{W} + (kb)^2 B(1+2\sin^2\theta_0)/2,$$
$$\mathscr{J}_1' \approx i(kb)^2 B\sin\theta_0,$$
$$\mathscr{J}_2' \approx (kb)^2 B/4;\quad \mathscr{J}_{2n}', \mathscr{J}_{2n-1}' = O(k^{2n}). \tag{68}$$

The next terms are two orders higher in k, and the result $\mathscr{J}_n' \to \mathscr{W}\delta_{n0}$ for $kb \to 0$ corresponds to small-spaced scatterers.

We construct $\mathscr{N}$ by using $-i\mathrm{S}_e$ in the first form of (59) or N_n in the second. For the second form, $\int_b^\infty pN_n[\ ]$, for $n=0$ and 1 we use $\int_0^\infty(p-1)N[\ ]$ because the corresponding integrals without p vanish; for $n>1$, we use $\int_b^\infty + \int_b^\infty(p-1)$. Correct to $O(w^2)$ and to leading terms in k for small kb

$$\begin{aligned}
&\mathscr{N}_0 \approx \mathfrak{R}_0 = -(2/\pi)I^0,\\
&I^0 = 2w\int_0^\infty(p-1)\ln\frac{2}{ckbu}\,du = I_0\ln\frac{2}{ckb} - I_\ell,\\
&c \approx 1.781;\\
&I_0 = 2w\int(p-1)du = -2w + w^2,\\
&I_\ell = 2w\int(p-1)\ln u\,du \approx 2w[1 + w(2\ln 2 - \tfrac{5}{4})];\\
&\mathscr{N}_1 \approx \mathfrak{R}_1 = i(2/\pi)I_0\sin\theta_0 = -i(2/\pi)w(2-w)\sin\theta_0.
\end{aligned} \tag{69}$$

If $w\sim 1$, then $I_\ell \sim \ln 2\pi$, and $I^0 \sim -\ln(4\pi/ckb)$. The corrections to $\mathscr{N}_0, \mathscr{N}_1$ are $O(k^2)$. For $n>2$, from

$$N_n(x) \approx -\frac{(n-1)!}{\pi}\left(\frac{2}{x}\right)^n,$$

we obtain the dominant terms for small kb

$$\begin{aligned}
&\mathscr{N}_{2n} \approx \frac{\mathfrak{R}_{2n}}{(kb)^{2n}} = -\frac{(2n-1)!}{\pi}\left(\frac{2}{kb}\right)^{2n} I_{2n},\\
&\mathscr{N}_{2n+1} \approx -i\mathscr{N}_{2n}4n\sin\theta_0,\quad I_{2n} = 2w\int_1^\infty\frac{p}{u^{2n}}du.
\end{aligned} \tag{70}$$

For small w, to $O(w^2)$,

$$I_2 = 2w[1 + w(1-\ln 2)] \approx 2w(1 + w0.307),$$
$$I_{2n} \approx \frac{2w}{2n-1}\left(1 + \frac{w}{2n-2}(2n-3+2^{-2n+2})\right);$$

the next term in I_2 equals $2w^3(1 - 6\ln 2 - 3\ln 3) \approx 2w^3 6.45$. For w near 1,

$$\frac{I_{2n}}{2w} \sim \zeta(2n)\left(1 - \frac{2n(1-w)}{w}\right) + n(2n+1)\frac{(1-w)^2}{w^2}[\zeta(2n) + \zeta(2n+1)].$$

If $w\to 1$, then the asymptotic result reduces to $\zeta(2n)$, the Riemann zeta function, as before for the periodic case.[3] For the dipole, $I_2/2 \sim \zeta(2) \approx \pi^2/6 \approx 1.645$.

B. Bounded obstacles

For bounded scatterers, we use series of spherical harmonics

$$\begin{aligned}
&g_{r0} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_n^m Y_n^m(\hat{\mathbf{r}}),\\
&a_n^m = a_n^m(\hat{\mathbf{r}}_0) = \sum_{\nu\mu} a_{n\nu}^{m\mu} Y_\nu^{-\mu}(\hat{\mathbf{r}}_0);\\
&G_{r0} = \sum_{nm} A_n^m Y_n^m(\hat{\mathbf{r}});\quad Y_n^m(\hat{\mathbf{r}}) = P_n^m(\cos\theta)e^{im\varphi},\\
&P_n^{-m} = P_n^m(-1)^m(n-m)!/(n+m)! = P_n^m C_n^m;\\
&a_{n\nu}^{m\mu} = (-1)^{n+\nu}a_{\nu m}^{-\mu-m}.
\end{aligned}$$

From (16), in terms of coefficients[14] d_ℓ of the expansion

$$Y_\nu^{-\mu}Y_s^t = \sum_\ell d_\ell Y_\ell^{t-\mu},$$

we have in general

$$A_n^m = a_n^m + \sum_{\nu\mu}\sum_{st} a_{n\nu}^{m\mu}A_s^t\sum_\ell d_\ell\left(\begin{matrix}-\mu\\ \nu\end{matrix}\middle|\begin{matrix}t\\ s\end{matrix}\right)\mathscr{H}_\ell^{t-\mu} \tag{71}$$

and for spheres

$$A_n^m = (-1)^m a_n\left[Y_n^{-m}(\hat{\mathbf{r}}_0) + \sum A_s^t\sum d_\ell\left(\begin{matrix}-m\\ n\end{matrix}\middle|\begin{matrix}t\\ s\end{matrix}\right)\mathscr{H}_\ell^{t-m}\right]. \tag{72}$$

These systems are the same as considered before[4] (p. 644 ff) for the doubly periodic planar lattice, but the present $\mathscr{H}$ represents distribution integrals. We have

$$\begin{aligned}
\mathscr{H}_n^m &= \mathrm{S}[Y_n^m(\hat{\mathbf{r}}_c) + Y_n^m(\hat{\mathbf{r}}_c')]\\
&= (-1)^m P_n^m(0)\rho\int d\mathbf{R}\,p(\mathbf{R})e^{i\mathbf{k}\cdot\mathbf{R}}i^n h_n(kR)e^{im\Phi},
\end{aligned} \tag{73}$$

with $h_n = h_n^{(1)} = j_n + in_n$ and the corresponding decomposition $\mathscr{H} = \mathscr{J} + i\mathscr{N}$ as before. Either form requires that $n-m$ be even: for odd parity,

$$Y_n^m(\hat{\mathbf{r}}) + Y_n^m(\hat{\mathbf{r}}') = Y_n^m(\hat{\mathbf{r}})[1 + (-1)^{n-m}]$$

vanishes, and so does $P_n^m(0)$. The second form follows from the first by using

$$\int_c e^{i\mathbf{k}_c\cdot\mathbf{r}}Y_n^m(\hat{\mathbf{r}}_c) = i^n Y_n^m(\hat{\mathbf{r}})h_n(kr)$$

and specializing to $\hat{\mathbf{r}}(\tfrac{1}{2}\pi, \pi+\Phi)$ to correspond to the displayed $e^{i\mathbf{k}\cdot\mathbf{R}}$. For negative m, we have $\mathscr{H}_n^{-m} = C_n^m\mathscr{H}_n^m(-\Phi)$.

No explicit analog of p of (62) exists for two-dimensional distributions, but for circular symmetry $p(R)$ can be computed numerically from the Percus–Yevick integral equa-

tion or from other approximations (hypernetted chain or Born–Green) or from machine simulations.[9] The virial expansion for identical impenetrable circular disks of diameter b analogous to (63) can be obtained solely from geometrical considerations; in terms of $u = R/b$ and $w = \pi b^2/4$, we again have $p = 0$ for $u < 1$, and $p = 1$ for $u > 2$, but now the first shell is specified by

$$p = 1 + \frac{8w}{\pi}\left\{\cos^{-1}\frac{u}{2} - \frac{u}{2}\left[1 - \left(\frac{u}{2}\right)^2\right]^{1/2}\right\}, \quad 1<u<2. \tag{74}$$

The structure factor for circular symmetry

$$\mathscr{W}(\mathscr{K}) = 1 + 8w\int_0^\infty (p-1)J_0(\mathscr{K}u)u\,du,$$
$$(\mathscr{K}/kb)^2 = (\alpha - \alpha_0)^2 + (\beta - \beta_0)^2$$
$$= \sin^2\theta + \sin^2\theta_0 - 2\sin\theta\sin\theta_0\cos(\varphi - \varphi_0) \tag{75}$$

may be evaluated numerically by means of integral-equation approximations[9] for p.

Using (74) and (75), we obtain for small $\mathscr{K}$,

$$\begin{aligned}\mathscr{W}(\mathscr{K}) &= 1 - 4w + \frac{12\sqrt{3}}{\pi}w^2 \\ &\quad + \frac{\mathscr{K}^2 w}{2}\left[1 - \frac{w8}{3}\left(1 + \frac{9\sqrt{3}}{4\pi}\right)\right] \\ &\quad + \mathscr{O}(w^3) + \mathscr{O}'(k^3) \\ &\approx 1 - 4w + 6.6159w^2 + \tfrac{1}{2}\mathscr{K}^2 w(1 - 5.9746w) \\ &\approx \mathscr{W} + \mathscr{K}^2 B.\end{aligned} \tag{76}$$

The first three terms of $\mathscr{W}(0) = \mathscr{W}$ in (76) differ from the corresponding terms of $\mathscr{W}_2$ in (48) in that $\mathscr{W}_2$ approximates the third by $7w^2$. A more complete expansion of $\mathscr{W}(\mathscr{K})$ would show the same structure as (65) with $\mathscr{W}(0) = \mathscr{W}$ as a factor. We use $\mathscr{W} \approx \mathscr{W}_2$ and $B \approx w(1 - 6w)/2$ in the following.

For large $\mathscr{K}$, we integrate (75) by parts and use the scaled-particle approximation for $p(b)$, and $J_1(x) \sim (2/\pi x)^{1/2}\sin(x - \pi/4)$, to obtain

$$\mathscr{W}(\mathscr{K}) \sim 1 - \frac{4w(2-w)(2/\pi)^{1/2}}{(1-w)^2}\frac{\sin(\mathscr{K} - \pi/4)}{\mathscr{K}^{3/2}}, \tag{77}$$

which tends to unity more rapidly with increasing $\mathscr{K}$ than the corresponding result in (66).

For aligned elliptic disks with principal diameters b_2 and b_3, we write

$$\mathbf{R} = u(\hat{\mathbf{x}}b_2\cos\tau + \hat{\mathbf{y}}b_3\sin\tau) \equiv \mathbf{R}(u)$$

with $u = 1$ as the exclusion ellipse, and use $d\mathbf{R} = b_2 b_3 u\,du\,d\tau$ to integrate over similar ellipses. For such elliptically symmetric statistics, we work with (74)–(77) in terms of

$$(\mathscr{K}/k)^2 = b_2^2(\alpha - \alpha_0)^2 + b_3^2(\beta - \beta_0)^2$$

and $w = \rho\pi b_2 b_3/4$ on the basis of geometrical considerations. The geometrical distribution of hard circular disks of diameter b, and variable $\mathbf{R} = ub\hat{\mathbf{R}}$ with $u = 1$ as the exclusion circle, determines the value $p(R) = p\{u\}$ corresponding to the equiprobability circle $\mathbf{R} = ub\hat{\mathbf{R}}$ with origin of u at the center of one disk regarded as fixed. A homogeneous deformation of space that leaves the origin fixed, a centro-affine transformation that converts the circular disks (b) to aligned elliptic disks $(b_2, b_3$ such that $b_2 b_3 = b^2)$, converts the equiprobability circle $\mathbf{R} = ub\hat{\mathbf{R}}$ around the origin to the corresponding ellipse $\mathbf{R}(u)$ on which we use $p\{u\}$.

For circular symmetry, the second form in (73) reduces to two sets corresponding to n,m both even or both odd:

$$\begin{aligned}\mathscr{H}_n^m &= P_n^m(0) i^{n-m} e^{im\varphi_0} 2\pi\rho \\ &\quad\times \int_b^\infty p(R) h_n(kR) J_m(kR\sin\theta_0) R\,dR,\\ P_{2n}^{2m}(0) &= (-1)^{n-m}(2n+2m-1)!!/2^{n-m}(n-m)!,\\ P_{2n+1}^{2m+1}(0) &= (2n+2m+1)P_{2n}^{2m}(0),\\ n!! &= n(n-2)!!, \quad 1!! = 0!! = 1.\end{aligned} \tag{78}$$

For normal incidence $(\theta_0 = 0)$, only

$$\mathscr{H}_{2n}^0 = P_{2n}^0(0)(-1)^n 2\pi\rho\int_b^\infty p(R) h_{2n}(kR) R\,dR \tag{79}$$

is required. More generally, for elliptically symmetric statistics, the symmetry properties of (73) are roughly similar to those of the lattice sums for the rectangular unit cell discussed before for (4:18), p., 646. Thus there are four sets of $\mathscr{H}$'s, two for even indices and two for odd:

$$\begin{aligned}\mathscr{H}_{2n}^{2m} &= (-1)^n P_{2n}^{2m}(0)\int d\mathbf{R}\, p h_{2n}(kR)[\cos X\cos Y\cos 2m\Phi \\ &\quad - i\sin X\sin Y\sin 2m\Phi],\\ \mathscr{H}_{2n+1}^{2m+1} &= (-1)^n P_{2n+1}^{2m+1}(0)\int d\mathbf{R} \\ &\quad\times p h_{2n+1}[\sin X\cos Y\cos(2m+1)\Phi \\ &\quad + i\cos X\sin Y\sin(2m+1)\Phi];\end{aligned} \tag{80}$$

$$X = k\alpha_0 R\cos\Phi = k\alpha_0 b_2 u\cos\tau,$$
$$Y = k\beta_0 R\sin\Phi = k\beta_0 b_3 u\sin\tau,$$

where

$$\int d\mathbf{R} = b_2 b_3\int_1^\infty du\, u\int_0^{2\pi} d\tau,$$

and

$$\int_0^{2\pi} d\tau = 4\int_0^{\pi/2} d\tau.$$

Subsequently, we use $\mathscr{H}^c$ and $\mathscr{H}^s$ to indicate components involving $\cos m\Phi$ and $\sin m\Phi$. For normal incidence $(\alpha_0 = \beta_0 = 0, X = Y = 0)$, the only set that arises is

$$\begin{aligned}\mathscr{H}_{2n}^{2m} &= (-1)^n P_{2n}^{2m}(0)\int d\mathbf{R} \\ &\quad\times p h_{2m}(kR)\cos 2m\Phi = \mathscr{H}_{2n}^{2mc}.\end{aligned} \tag{81}$$

In (80) and (81) we use

$$R = u(b_2^2\cos^2\tau + b_3^2\sin^2\tau)^{1/2}$$

and

$$e^{i\Phi} = u(b_2\cos\tau + ib_3\sin\tau)/R.$$

Essentially as for (67), for all cases with $n - m$ even, we have

$$\mathscr{F}_n^m = 2CY_n^m(\hat{\mathbf{r}}_0) + \mathscr{F}_n^{\prime m} - \delta_{n0}$$

$$\mathscr{F}_n^{\prime m} = \mathscr{M}\mathscr{W}[K]Y_n^m(\hat{\mathbf{r}}). \tag{82}$$

The specular contribution in $C = \tau\rho/k^2\cos\theta_0$ is the same as the result for the doubly periodic array[4] with ρ^{-1} replacing the area of unit cell. For small kb and circular statistics, from (82) in terms of $\mathscr{W}[K]$ of (76), or from the form $\mathscr{F}(j)$ with $j_n \approx x^n/(2n+1)!!$,

$$\mathscr{F}_0^{\prime 0} \approx \mathscr{W} + (kb)^2 B\left(\tfrac{2}{3} + \sin^2\theta_0\right),$$

$$\mathscr{F}_1^{\prime 1} \approx -(kb)^2 B\tfrac{2}{3}\sin\theta_0 e^{i\varphi_0},$$

$$\mathscr{F}_2^{\prime 0} \approx -(kb)^2 B\tfrac{2}{15}; \quad \mathscr{F}_{2n}^{\prime 0}, \mathscr{F}_{2n-1}^{\prime 1} = \mathscr{O}(k^{2n}). \tag{83}$$

The next terms are two orders higher in kb.

For elliptic statistics, from (82)with

$$(\mathscr{K}/k)^2 = b_2^2(\alpha - \alpha_0)^2 + b_3^2(\beta - \beta_0)^2,$$

or from (74) and (80) in terms of u and τ,

$$\mathscr{F}_{00}^{\prime} - \mathscr{W} \approx Bk^2\left[b_2^2\left(\tfrac{1}{3} + \alpha_0^2\right) + b_3^2\left(\tfrac{1}{3} + \beta_0^2\right)\right],$$

$$\mathscr{F}_1^{\prime 1} = -Bk^2\tfrac{2}{3}(b_2^2\alpha_0 + ib_3^2\beta_0),$$

$$\mathscr{F}_2^{\prime 0} \approx -(Bk^2/15)(b_2^2 + b_3^2),$$

$$\mathscr{F}_2^{\prime 2} \approx \mathscr{F}_2^{\prime 2c} \approx \tfrac{2}{5}(Bk^2)(b_2^2 - b_3^2),$$

$$\mathscr{F}_2^{\prime 2s} = \mathscr{O}(k^4), \tag{84}$$

where the corrections are $\mathscr{O}(k^4)$. If $b_2 = b_3$, these results reduce to the corresponding ones in (83). For both sets, if $k \to 0$, then $\mathscr{F}_n^{\prime m} \to \mathscr{W}\delta_{n0}$ for small-spaced scatterers.

We obtain $\mathscr{N}$ by using $-i\mathrm{S}_e$ in the first form in (73) or n_n in the second. Corresponding to (70), the dominant terms of $\mathscr{N}$ for small k may be obtained by using $n_m(x) \approx -(2n-1)!!/x^{n+1}$ in the $\mathscr{N}$ form of (80). Proceeding essentially as before for the periodic analog (4:90), p. 653, we obtain in terms of $t = b_3/b_2$,

$$\mathscr{N}_{2n}^{2m} \approx \frac{\mathfrak{N}_{2n}^{2m}}{(kb_2)^{2n+1}} = -\frac{(4n-1)!!!(-1)^n P_{2n}^{2m}(0)}{(kb_2)^{2n+1}} I_{2n} M_{2n}^{2m}(t),$$

$$\mathscr{N}_{2n+1}^{2m+1} \approx \frac{\mathfrak{N}_{2n+1}^{2m+1}}{(kb_2)^{2n+1}} = -\frac{(4n+1)!!!(-1)^n P_{2n+1}^{2m+1}(0)}{(kb_2)^{2n+1}}$$

$$\times I_{2n}\frac{\sin\theta_0}{2}(M_{2n}^{2m}e^{i\varphi_0} + M_{2n}^{2m+2}e^{-i\varphi_0}), \tag{85}$$

$$I_0 = 8w\int_0^\infty (p-1)du, \quad I_{2n} = 8w\int_1^\infty \frac{p}{u^{2n}}du,$$

$$M_{2n}^{2m}(t) = \frac{2}{\pi}\int_0^{\pi/2} d\tau \frac{\mathrm{Re}(\cos\tau + it\sin\tau)^{2m}}{(\cos^2\tau + t^2\sin^2\tau)^{n+m+1/2}},$$

where the M's may be expressed in terms of elliptic integrals. The present IM plays the same role as the periodic[4] L; as before we need consider only $t > 1$ explicitly, and obtain results for $t < 1$ by using

$$M_{2n}^{2m}(t^{-1}) = (-1)^m t^{2n+1} M_{2n}^{2m}(t).$$

For $n = 0$ and 1, in terms of p of (75),

$$-\mathscr{N}_0^0 kb_2 \approx -\mathfrak{N}_0^0 \equiv I_0 M_0^0; \quad I_0 \approx -8w(1 - B_1 w),$$

$$B_1 = 8\left(\frac{3\sqrt{3}}{\pi 4} - \frac{1}{3}\right) \approx 0.6413,$$

$$M_0^0 = (2/\pi t)K(\nu), \quad \nu = 1 - t^{-2};$$

$$-\mathscr{N}_1^1 kb_2 \approx -\mathfrak{N}_1^1 \equiv I_0\sin\theta_0(M_0^0 e^{i\varphi_0} + M_0^2 e^{-i\varphi_0})/2,$$

$$M_0^2 = (2/\pi t\nu)[2E - (2-\nu)K]; \tag{86}$$

$$E(\nu) = \int_0^{\pi/2}(1 - \nu\sin^2\tau)^{1/2}d\tau,$$

$$K(\nu) = \int_0^{\pi/2}(1 - \nu\sin^2\tau)^{-1/2}d\tau,$$

where E and K are the tabulated complete elliptic integrals, and the next terms are $\mathscr{O}(k)$. The dominant contributions for $n > 1$ involve

$$I_{2n} \approx \frac{8w}{2n-1}\{1 + wB_2 - wI_{2n}^{\prime}\},$$

$$B_2 = \frac{8}{3}\left(1 - \frac{3\sqrt{3}}{4\pi}\right) \approx 1.5640,$$

$$I_{2n}^{\prime} = \frac{8}{\pi 2^{2n-2}}\int_{1/2}^1 \frac{(1-u^2)^{1/2}}{u^{2n-1}}du,$$

$$I_2^{\prime} = \frac{8}{\pi}\left(L - \frac{1}{2}\sqrt{3}\right), \quad I_4^{\prime} = \frac{2}{\pi}\left(\sqrt{3} - \frac{1}{2}L\right),$$

$$I_6^{\prime} = \frac{1}{2\pi}\left(\frac{7\sqrt{3}}{4} - \frac{1}{8}L\right); \quad L = \ln(2 + \sqrt{3}) \approx 1.317.$$

The dipole terms are given by

$$\mathscr{N}_2^0(kb_2)^3/3 \to \mathfrak{N}_2^0 2/3 = -I_2 M_2^0,$$

$$\mathscr{N}_2^2(kb_2)^3/9 \to \mathfrak{N}_2^2/9 = I_2 M_2^2,$$

$$I_2 = 8w(1 + wB_2 - wI_2) \approx 8w(1 + 0.4157w),$$

$$M_2^0 = (2/\pi t)E, \quad M_2^2 = (2/\pi 3t\nu)[(2-\nu)E - 2(1-\nu)K]. \tag{87}$$

If $t \to 1$ for circular symmetry, then M_0^2 and $M_2^2 \to 0$, and M_0^0 and $M_2^0 \to 1$; the leading terms of $\mathscr{N}_2^2$ is then $\mathscr{O}(k^{-1})$. The dominant terms for circular symmetry are the subsets

$$\mathscr{N}_{2n}^0 \approx -\frac{(2n-1)!!!(4n-1)!!}{2^n n!(kb)^{2n+1}} I_{2n},$$

$$\mathscr{N}_{2n+1}^1 \approx -\frac{(2n+1)!!!(4n+1)!!}{2^{n+1} n!(kb)^{2n+1}} I_{2n}\sin\theta_0 e^{i\varphi_0}, \tag{88}$$

as obtained directly from the form (78).

Our consideration of $\mathscr{W}(\mathscr{K})$ and $\mathscr{H}_n^m$ for elliptic symmetry based on $p(\mathbf{R}) = p\{u\}$, with $p\{u\}$ as the values of the radial distribution function for circular symmetry, have led to approximations in terms of elliptic integrals which make explicit the departures from maximal symmetry. By integrating over symmetrical ellipses, we isolated integrals of $p\{u\}$ over u that are identical with results for circles, so that existing approximations and computations[9] for circles can be applied directly.

In distinction to the one-dimensional distribution, for which the periodic case is a proper limit of p, there are no comparable representations or approximations for a two-dimensional distribution function $p(\mathbf{R})$. There is no unique pe-

riodic limit in two dimensions, and a great deal more structure (noncentral potentials) would have to be introduced, and more than geometrical arguments would be required, to trace the evolution of a gas of rigid disks through the liquid to some specific periodic limit. For distributions governed solely by the geometry of the exclusion region, a realistic bound corresponds to the two-dimensional analog of an amorphous solid; instead of using the close-packed value $w_s \approx 0.785$ for a square array or $w_h \approx 0.907$ for a hexagonal array, we use the experimental value $w_d \approx 0.84$ for the densest random packing (which approximates the mean of w_s and w_h). Nevertheless, the earlier results for the rectangular lattice[4] with the square as a special case provide rough (less symmetrical) analogs of the present forms for the ellipse and circle, as well as estimates for bounds for the present case.

In the explicit low-frequency approximations of $\mathscr{H}_n^m$ for the rectangular array given by (4:14), p. 659 ff, the function $L(t)$ corresponds to the present $IM(t)$. For the square lattice,

$$L_0^0(1) \approx -3.900, \quad L_2^0(1) \approx 9.03, \quad L_2^2(1) = 0;$$

and for rectangles

$$L_0^0(2) \approx -2.521, \quad L_2^0(2) \approx 4.049, \quad L_2^2(2) \approx 1.856;$$
$$L_0^0(4) \approx -1.135, \quad L_2^0(4) \approx 2.815, \quad L_2^2(4) \approx 2.267.$$

Normalizing $L_n^m(t)$ by dividing by $L_n^0(1)$, we write

$$m_0^0(2) \approx 0.646, \quad m_2^0(2) = 0.448, \quad m_2^2(2) = 0.206;$$
$$m_0^0(4) \approx 0.291, \quad m_2^2(4) = 0.312, \quad m_2^2(4) \approx 0.251.$$

The corresponding values for the present problem are $M_n^0(1) = 1, M_2^2(1) = 0$, and

$$M_0^0(2) \approx 0.686, \quad M_2^0(2) \approx 0.385, \quad M_2^2(2) \approx 0.0616;$$
$$M_0^0(4) \approx 0.446, \quad M_2^0(4) \approx 0.171, \quad M_4^2(4) \approx 0.0447. \tag{89}$$

The normalized monopole terms are larger for the ellipse than for the rectangle, and the normalized dipole terms are smaller, i.e., $M_0(t) > m_0(t)$ and $M_2(t) < m_2(t)$. The ratios M_2^2/M_2^0 are smaller than m_2^2/m_2^0; with increasing t, the first tends to $\frac{1}{3}$ and the second to 1.

VI. CLOSED-FORM APPROXIMATIONS

The algebraic systems (57) and (71) may be expanded in series form, truncated and solved in closed form, or investigated numerically. We illustrate truncation by retaining only the monopole and dipole terms for elliptical scatterers and statistics. The results are applied to obtain low-frequency approximations for small-spaced scatterers with emphasis on dominant multipole coupling effects.

The isolated scattering amplitude has the form

$$g_{r0} = a_0 + \hat{\mathbf{r}}\cdot\bar{\mathbf{a}}\cdot\hat{\mathbf{k}} = a_0 + \sum_{i=1}^{n} a_i \zeta_i \zeta_i^0$$
$$= a_0 + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\gamma_i\gamma_j^0;$$
$$\gamma_1 = \cos\theta, \quad \gamma_2 = \sin\theta\cos\varphi, \quad \gamma_3 = \sin\theta\sin\varphi; \tag{90}$$

where $\bar{\mathbf{a}}$ is a symmetrical dyadic with principal values a_i along $\hat{\zeta}_i$, and $n = 3$ or 2 for ellipsoids or elliptic cylinders.

For the corresponding multiple scattering amplitude, we write

$$G_{r0} = A_0 + \hat{\mathbf{r}}\cdot\mathbf{A} = A_0 + \sum_{i=1}^{n} A_i\gamma_i = \sum_{i=0}^{n} A_i\gamma_i, \quad \gamma_0 = 1. \tag{91}$$

We work explicitly with $n = 3$, and set γ_3 and $\varphi = 0$ for $n = 2$.

From the algebraic systems (57) and (71), or from the functional equation (16), we obtain

$$A_0 = a_0(1 + \mathscr{H}_{00}A_0 + \mathscr{H}\cdot\mathbf{A}), \quad \mathbf{A} = \bar{\mathbf{a}}\cdot(\hat{\mathbf{k}} + \mathscr{H}A_0 + \bar{\mathscr{H}}\cdot\mathbf{A});$$
$$\mathscr{H} = \hat{\mathbf{x}}\mathscr{H}_{02} + \hat{\mathbf{y}}\mathscr{H}_{03}, \tag{92}$$
$$\bar{\mathscr{H}} = \mathscr{H}_{11}\hat{\mathbf{z}}\hat{\mathbf{z}} + \mathscr{H}_{22}\hat{\mathbf{x}}\hat{\mathbf{x}} + \mathscr{H}_{23}\hat{\mathbf{x}}\hat{\mathbf{y}} + \mathscr{H}_{32}\hat{\mathbf{y}}\hat{\mathbf{x}} + \mathscr{H}_{33}\hat{\mathbf{y}}\hat{\mathbf{y}},$$

where the $\mathscr{H}_{ij} = \mathscr{H}_{ji}$ are linear combinations of the distribution integrals we considered. For the planar array of cylinders, in terms of (59) ff,

$$\mathscr{H}_{00} = \mathscr{H}_0, \quad \mathscr{H}_{02} = -i\mathscr{H}_1, \quad \mathscr{H}_{11} = \tfrac{1}{2}(\mathscr{H}_0 + \mathscr{H}_2),$$
$$\mathscr{H}_{22} = \tfrac{1}{2}(\mathscr{H}_0 - \mathscr{H}_2). \tag{93}$$

For bounded obstacles, in terms of (73) ff,

$$\mathscr{H}_{00} = \mathscr{H}_0^0, \quad \mathscr{H}_{02} = \mathscr{H}_1^{1c}, \quad \mathscr{H}_{03} = \mathscr{H}_1^{1s},$$
$$\mathscr{H}_{11} = \tfrac{1}{3}(\mathscr{H}_0^0 + 2\mathscr{H}_2^0),$$
$$\mathscr{H}_{23} = \tfrac{1}{6}\mathscr{H}_2^{2s}, \quad \mathscr{H}_{22} = \tfrac{1}{3}(\mathscr{H}_0^0 - \mathscr{H}_2^0) + \tfrac{1}{6}\mathscr{H}_2^{2c},$$
$$\mathscr{H}_{33} = \tfrac{1}{3}(\mathscr{H}_0^0 - \mathscr{H}_2^0) - \tfrac{1}{6}\mathscr{H}_2^{2c}. \tag{94}$$

The superscripts c and s correspond to terms in $\cos m\Phi$ and $\sin m\Phi$ as indicated after (80). Thus, in general, the four A's are specified by four simultaneous equations in terms of the a's and $\mathscr{H}$'s. First we consider the simpler systems that arise for several special cases in order to display the structure of the multipole coupling; then we obtain closed low-frequency forms of (92).

If the principal axes of the dipole are along x,y,z (the symmetrical monolayer), then (92) reduces to two sets.

$$A_1 = a_1(\gamma_1^0 + \mathscr{H}_{11}A_1); \quad A_i = a_i(\gamma_i^0 + \sum_{j\neq 1}\mathscr{H}_{ij}A_j);$$
$$i \text{ or } j = 0,2,3, \tag{95}$$

corresponding, respectively, to the decomposition $G = G_A + G_S$ into components antisymmetrical and symmetrical with respect to the plane of centers ($z = 0$). Introducing the self-coupling coefficients,

$$\mathscr{A}_n = a_n/(1 - a_n\mathscr{H}_{nn}),$$

we write the antisymmetrical component as

$$A_1 = a_1\gamma_1^0/(1 - a_1\mathscr{H}_{11}) = \mathscr{A}_1\gamma_1^0,$$
$$G_A(\hat{\mathbf{r}}.\hat{\mathbf{k}}) = A_1\gamma_1 = \mathscr{A}_1\gamma_1\gamma_1^0, \tag{96}$$

and specify the symmetrical component $G_S = A_0 + A_2\gamma_2 + A_3\gamma_3$ by

$$A_0 = \mathscr{A}_0(1 + A_1\mathscr{H}_{02} + A_3\mathscr{H}_{03}),$$
$$A_2 = \mathscr{A}_2(\gamma_2^0 + A_0\mathscr{H}_{20} + A_3\mathscr{H}_{23}),$$
$$A_3 = \mathscr{A}_3(\gamma_3^0 + A_0\mathscr{H}_{30} + A_2\mathscr{H}_{32}). \tag{97}$$

For normal incidence, $\gamma_2^0 = \gamma_3^0 = 0$, and the displayed lattice, sums vanish; they are not in the subset (81). Then $G_S = \mathscr{A}_0$ and $G = \mathscr{A}_0 + \mathscr{A}_1\gamma_1\gamma_1^0$.

For incidence in the zx plane, only the $\mathscr{H}_n^{mc}$ arise and only $\mathscr{H}_{02} = \mathscr{H}_1^{1c}$ survives in (97). Thus $G_S = A_0 + A_2\gamma_2$ in terms of

$$A_0 = \mathscr{A}_0(1 + A_2\mathscr{H}_{02}), \quad A_2 = \mathscr{A}_2(\gamma_2^0 + A_0\mathscr{H}_{20});$$
$$A_0 = \mathscr{A}_0(1 + \mathscr{H}_{02}\mathscr{A}_2\gamma_2^0)/D,$$
$$A_2 = \mathscr{A}_2(\gamma_2^0 + \mathscr{H}_{20}\mathscr{A}_0)/D, \quad D = 1 - \mathscr{A}_0\mathscr{A}_2(\mathscr{H}_{20})^2,$$

and the multiple scattered amplitude equals

$$G(\hat{\mathbf{r}},\hat{\mathbf{k}}) = G_A + G_S = \mathscr{A}_1\gamma_1\gamma_1^0 + \frac{\mathscr{A}_0 + \mathscr{A}_2\gamma_2\gamma_2^0 + \mathscr{A}_0\mathscr{A}_2\mathscr{H}_{02}(\gamma_2 + \gamma_2^0)}{1 - \mathscr{A}_0\mathscr{A}_2(\mathscr{H}_{02})^2},$$

$$\mathscr{A}_i = a_i/(1 - a_i\mathscr{H}_{ii}). \tag{98}$$

Here the $\mathscr{A}_0\mathscr{A}_2$ terms correspond to monopole–dipole interactions. This same structure also arises for spheroids with symmetry axis along $\hat{\mathbf{z}}$ and arbitrary $\hat{\mathbf{k}}$, and for the two-dimensional problem ($\gamma_2 = \sin\theta$) of elliptic cylinders.

Although the general case in (97) is a simple system that can be solved directly, in order to delineate cross coupling between $\mathscr{A}_2$ and $\mathscr{A}_3$, we first incorporate such effects in coefficients $\mathscr{B}, \mathscr{C}$. We have

$$A_2 = \mathscr{B}_{23} + A_0\mathscr{C}_{23}, \quad A_3 = \mathscr{B}_{32} + A_0\mathscr{C}_{32},$$
$$\mathscr{B}_{23} = \mathscr{A}_2(\gamma_2^0 + \gamma_3^0\mathscr{A}_3\mathscr{H}_{23})/D,$$
$$\mathscr{C}_{23} = \mathscr{A}_2(\mathscr{H}_{20} + \mathscr{A}_3\mathscr{H}_{23}\mathscr{H}_{30})/D,$$
$$D = 1 - \mathscr{A}_2\mathscr{A}_3(\mathscr{H}_{23})^2, \tag{99}$$

and $\mathscr{B}_{32}, \mathscr{C}_{32}$ follow on interchanging 2 and 3. If the monopole is negligible (e.g., if the relative compressibility of the particles is unity), then $A_2 = \mathscr{B}_{23}$ and $A_3 = \mathscr{B}_{32}$. From (99) and (97).

$$A_0 = \frac{\mathscr{A}_0(1 + \mathscr{H}_{02}\mathscr{B}_{23} + \mathscr{H}_{03}\mathscr{B}_{23})}{1 - \mathscr{A}_0(\mathscr{H}_{02}\mathscr{C}_{23} + \mathscr{H}_{03}\mathscr{C}_{32})} \tag{100}$$

and A_2 and A_3 follow from (99).

An alternative simplification of (92) is obtained by specializing the forms for arbitrary alignment to $\hat{\boldsymbol{\xi}}_3 = \hat{\mathbf{y}}$ and normal incidence $\hat{\mathbf{k}} = \hat{\mathbf{z}}$. Only the $\mathscr{H}_{ii}$ are nonvanishing, and from

$$A_0 = a_0(1 + A_0\mathscr{H}_{00}),$$
$$A_1 = a_{11}(1 + \mathscr{H}_{11}A_1) + a_{12}\mathscr{H}_{22}A_2,$$
$$A_2 = a_{21}(1 + \mathscr{H}_{11}A_1) + a_{22}\mathscr{H}_{22}A_2, \tag{101}$$

we have

$$A_0 = \mathscr{A}_0, \quad A_1 = [a_{11} - (a_{11}a_{22} - a_{12}^2)\mathscr{H}_{22}]/D,$$
$$A_2 = a_{21}/D;$$
$$D = 1 - a_{11}\mathscr{H}_{11} - a_{22}\mathscr{H}_{22} + (a_{11}a_{22} - a_{12}^2)\mathscr{H}_{11}\mathscr{H}_{22};$$
$$G = A_0 + A_1\gamma_1 + A_2\gamma_2. \tag{102}$$

the same results hold for the two-dimensional problem. Indicating orientation by $\hat{\boldsymbol{\xi}}_1 = \hat{\mathbf{z}}\cos\beta - x\sin\beta$, we rewrite the dipole terms as

$$A_1 = (a_1\cos^2\beta + a_2\sin^2\beta - a_1a_2\mathscr{H}_{22})/D,$$
$$A_2 = -(a_1 - a_2)\sin\beta\cos\beta/D,$$
$$D = 1 - a_1(\mathscr{H}_{11}\cos^2\beta + \mathscr{H}_{22}\sin^2\beta) - a_2(\mathscr{H}_{11}\sin^2\beta + \mathscr{H}_{22}\cos^2\beta) + a_1a_2\mathscr{H}_{11}\mathscr{H}_{22}. \tag{103}$$

The forward scattering amplitude for this case, $G_{zz} = A_0 + A_1$ with $\mathscr{H}_{ii} = \mathscr{H}_{ii}(\hat{\mathbf{z}})$, is substantially different from G of (98) evaluated for $\hat{\mathbf{r}} = \hat{\mathbf{k}} = \hat{\mathbf{z}}\cos\beta + \hat{\mathbf{x}}\sin\beta$, i.e., from $G_{kk} = A_0 + A_1\cos^2\beta + A_2\sin^2\beta$ with $\mathscr{H}_{ij} = \mathscr{H}_{ij}(\hat{\mathbf{k}})$. Although the orientation of a single ellipsoid with respect to an incident ray is the same in both cases, the orientation of the array is not, so that the coupling processes differ. In general (103) is considerably more complicated because it is much less symmetrical.

More generally, we reduce (92) to the form $G[g]$ of (32) which we have already analyzed, and show the relationships of g, g', g'', and $\mathscr{g}$ explicitly. Substituting the first form of g of (90) and the analog for g' into (36), we obtain

$$g'_{o} = a'_0 + \hat{\mathbf{r}}\cdot\tilde{\mathbf{a}}'\cdot\hat{\mathbf{k}}, \quad a'_0 = (1 + a_0)^{-1}a_0,$$
$$\tilde{\mathbf{a}}' = (\tilde{\mathbf{I}} + \tilde{\mathbf{a}}/n)^{-1}\cdot\tilde{\mathbf{a}}. \tag{104}$$

The amplitude g' for the radiationless obstacle, and the coefficients a' are imaginary for lossless scatterers. In practice, we start with g' or an approximation (derived, e.g., from perturbation of potential theory results) and use the inverse relations $a_0 = a'_0(1 - a'_0)^{-1}, \tilde{\mathbf{a}} = \tilde{\mathbf{a}}'\cdot(I - \tilde{\mathbf{a}}'/n)^{-1}$ to construct g. To analyze (92), we use (104) and the decomposition

$$\mathscr{H}_{ij} = \mathscr{J}_{ij}^0 - m_i\delta_{ij} + \mathscr{H}'_{ij}, \quad \mathscr{J}_{ij}^0 = 2C\gamma_i^0\gamma_j^0,$$
$$\mathscr{H}'_{ij} = \mathscr{J}'_{ij} + i\mathscr{N}_{ij}, \tag{105}$$

where $m_0 = 1$, and $m_i = 1/n$ with $n = 2$ or 3 if $i \neq 0$.

Substituting (105) into A_0 of (92), we make the specular contribution explicit

$$A_0 = a_0[1 + (2C - 1 + \mathscr{H}'_{00})A_0 + 2C(\gamma_2^0A_2 + \gamma_3^0A_3) + \mathscr{H}\cdot\mathbf{A}]$$

and then suppress $-1 + \mathscr{H}'_{00}$ by the form

$$A_0 = \mathscr{A}'_0\{[1 + 2CA_0 + C(\hat{\mathbf{k}} + \hat{\mathbf{k}}')\cdot\mathbf{A}] + \mathscr{H}\cdot\mathbf{A}\}$$
$$= A'_0\{[(1 + CG_{00}) + CG_{0'0}] + \mathscr{H}\cdot\mathbf{A}\}, \tag{106}$$
$$\mathscr{A}'_0 = \frac{a_0}{1 - a_0(-1 + \mathscr{H}'_{00})} = \frac{a'_0}{1 - a'_0(\mathscr{J}'_{00} + i\mathscr{N}_{00})} = \frac{a''_0}{1 - a''_0\mathscr{J}'_{00}}.$$

Here $a''_0 = a'_0/(1 - a'_0 i\mathscr{N}_{00}) \equiv a'_0\mathscr{D}_0$, such that $\mathscr{D}_0$ is real when a'_0 is imaginary, is the monopole coefficient of the radiationless distribution amplitude g''.

Similarly for $\mathbf{A}$ of (92), we incorporate $-\delta_{ij}/n$ directly, and make explicit the specular $\mathscr{J}^0$ terms,

$$\mathbf{A} = \tilde{\mathbf{a}}'\cdot[\hat{\mathbf{k}} + C(\hat{\mathbf{k}} + \hat{\mathbf{k}}')A_0 + C(\hat{\mathbf{k}}\hat{\mathbf{k}} + \hat{\mathbf{k}}'\hat{\mathbf{k}}')\cdot\mathbf{A}] + \tilde{\mathbf{a}}'\cdot(\mathscr{H}A_0 + \tilde{\mathscr{H}}\cdot\mathbf{A})$$

and then suppress $\tilde{\mathscr{H}}$ to obtain

$$\mathbf{A} = \tilde{\mathscr{A}}\cdot\{[\hat{\mathbf{k}}(1 + CG_{00}) + \hat{\mathbf{k}}'CG_{0'0}] + \mathscr{H}'A_0\},$$
$$\tilde{\mathscr{A}} = (\tilde{\mathbf{I}} - \tilde{\mathbf{a}}'\cdot\mathscr{H}')^{-1}\cdot\tilde{\mathbf{a}}' = (\tilde{\mathbf{I}} - \tilde{\mathbf{a}}''\cdot\tilde{\mathscr{F}}')^{-1}\cdot\tilde{\mathbf{a}}'', \qquad (107)$$

where $\tilde{\mathbf{a}}'' = (\tilde{\mathbf{I}} - i\tilde{\mathbf{a}}'\cdot\tilde{\mathscr{N}})^{-1}\cdot\tilde{\mathbf{a}}'$ is the dipole coefficient of the radiationless g''. The brackets in (106) enclose $(\langle\Psi_>\rangle + \langle\Psi_<\rangle)/2$ evaluated at $z=0$, i.e. the mean coherent field (excess pressure); and similarly the brackets in (107) enclose $\nabla(\langle\Psi_>\rangle + \langle\Psi_<\rangle)/2ik$ evaluated at $z=0$, the mean gradient (normalized velocity).

We eliminate the cross terms in (106) and (107) corresponding to monopole–dipole coupling, and obtain in terms of $L = 1 + CG_{00}$, $M = CG_{0'0}$,

$$A_0 = (a_0 + a\cdot\hat{\mathbf{k}})L + (a_0 + a\cdot\hat{\mathbf{k}}')M,$$
$$\mathbf{A} = (a + \tilde{a}\cdot\hat{\mathbf{k}})L + (a + \tilde{a}\cdot\hat{\mathbf{k}}')M,$$
$$a_0 = (1 - \mathscr{A}_0\mathscr{H}\cdot\tilde{\mathscr{A}}\cdot\mathscr{H})^{-1}\mathscr{A}'_0,$$
$$\tilde{a} = \tilde{\mathscr{A}}\cdot(I - a_0\mathscr{H}\mathscr{H}\cdot\tilde{\mathscr{A}}),$$
$$a = \mathscr{H}\cdot\tilde{\mathscr{A}}a_0 = \tilde{\mathscr{A}}\cdot\mathscr{H}a_0, \qquad (108)$$

as well as $\tilde{a} = (\tilde{\mathbf{I}} - \mathscr{A}'_0\tilde{\mathscr{A}}\cdot\mathscr{H}\mathscr{H})^{-1}\cdot\tilde{\mathscr{A}}$ and $a = \tilde{a}\cdot\mathscr{H}\mathscr{A}'_0$. Substituting into $G_{r0} = A_0 + \hat{\mathbf{r}}\cdot\mathbf{A}$ yields $G_{r0} = \mathscr{g}_{r0}L + \mathscr{g}_{r0'}M$ as in (32) with

$$\mathscr{g}_{r1} = a_0 + a\cdot(\hat{\mathbf{r}} + \hat{\mathbf{r}}_1) + \hat{\mathbf{r}}\cdot\tilde{a}\cdot\hat{\mathbf{r}}_1; \quad \hat{\mathbf{r}}_1 = \hat{\mathbf{k}},\hat{\mathbf{k}}'. \qquad (109)$$

The three coefficients a_0, a, and $\tilde{a}$ hold for both $\hat{\mathbf{k}}$ and $\hat{\mathbf{k}}'$ because only their tangential (x,y) components are involved. If we replace $\hat{\mathbf{k}}$ by $-\hat{\mathbf{k}}$, then a_0 and $\tilde{a}$ are unaltered, but $a(-\hat{\mathbf{k}}) = -a(\hat{\mathbf{k}})$. The equivalent scatterer $\mathscr{g}$, from which we obtain G by incorporating the specular effects via (32), satisfies the same reciprocity relation as G, i.e., $\mathscr{g}(\hat{\mathbf{r}}_1,\hat{\mathbf{r}}_2) = \mathscr{g}(-\hat{\mathbf{r}}_2, -\hat{\mathbf{r}}_1)$ with either $\hat{\mathbf{r}}_1$ or $\hat{\mathbf{r}}_2$ equaling either $\hat{\mathbf{k}}$ or $\hat{\mathbf{k}}'$. Substituting $\mathscr{g}_{r0}$ and $\mathscr{g}_{r0'}$ of (109) into (39), (42), ff, provides explicit illustrations of the general considerations.

The procedure (106)–(109) serves to delineate the relations between the various coefficients and amplitudes, and to display the role of the coherent field in the monopole A_0 and of its gradient in the dipole $\mathbf{A}$. However, we derive $\mathscr{g}$ more directly from (32) by substituting $\mathscr{g}_{r1} = \mathfrak{U}_0 + \hat{\mathbf{r}}\cdot\mathfrak{U}$ and $g_{r1} = a_0 + \hat{\mathbf{r}}\cdot\tilde{\mathbf{a}}\cdot\hat{\mathbf{r}}_1$, and denoting the corresponding distribution integrals generated by $S^{(0)}$ as $\mathscr{H}^{(0)} = \mathscr{H} - \mathscr{J}^0$. We obtain (92) in terms of $\mathfrak{U}_0,\mathfrak{U},\mathscr{H}^{(0)}$, and corresponding analogs of (106) and (107) devoid of specular contributions (no terms in C). This leads directly to $\mathfrak{U}_0 = \mathscr{A}'_0(1 + \mathscr{H}\cdot\mathfrak{U})$ and $\mathfrak{U} = \tilde{\mathscr{A}}\cdot(\hat{\mathbf{r}}_1 + \mathscr{H}\mathfrak{U}_0)$, and after eliminating the cross terms, to $\mathfrak{U}_0 = a_0 + a\cdot\hat{\mathbf{r}}_1$ and $\mathfrak{U} = a + \tilde{a}\cdot\hat{\mathbf{r}}_1$ with coefficients as in (108).

For small-spaced scatterers, and a'_0 and $\tilde{\mathbf{a}}'$ of order k^n, we use

$$\mathscr{J}'_{ij} \to \mathscr{W}m_i\delta_{ij}, \quad \mathscr{N}_{ij} \approx \mathscr{N}_{ii}\delta_{ij}, \qquad (110)$$

where $\mathscr{N}_{ii}$, obtained from (70) and (85) via (93) and (94), is of order k^{-n} for $i>0$, and $\mathscr{N}_{00}k^n \to 0$ as $k\to 0$. Thus

$$\mathscr{g}_{r1} \approx a_0 + \hat{\mathbf{r}}\cdot\tilde{a}\cdot\hat{\mathbf{r}}_1; \quad a_0 \approx a'_0/(1 - a'_0\mathscr{W}),$$
$$\tilde{a} = [\tilde{\mathbf{I}} - \tilde{\mathbf{a}}'(\mathscr{W}/n) - i\tilde{\mathbf{a}}'\tilde{\mathscr{N}}]^{-1}\cdot\tilde{\mathbf{a}}'$$
$$= (I - \tilde{\mathbf{a}}''\mathscr{W}/n)^{-1}\cdot\tilde{\mathbf{a}}'' \approx \tilde{\mathbf{a}}'' + \tilde{\mathbf{a}}''\cdot\tilde{\mathbf{a}}''\mathscr{W}/n, \qquad (111)$$
$$\tilde{\mathbf{a}}'' = (\tilde{\mathbf{I}} - i\tilde{\mathbf{a}}'\tilde{\mathscr{N}})^{-1}\cdot\tilde{\mathbf{a}}' = \tilde{\mathscr{D}}^{-1}\cdot\tilde{\mathbf{a}}'.$$

For lossless particles, $\tilde{\mathbf{a}}'$ and $\tilde{\mathbf{a}}''$ are imaginary, i.e., $\tilde{\mathscr{D}} = \tilde{\mathbf{I}} - i\tilde{\mathbf{a}}'\tilde{\mathscr{N}}$ is real; the factor $\tilde{\mathscr{D}}^{-1}$ introduces a k-independent coupling correction to the stripped dipole coefficient $\tilde{\mathbf{a}}'$. The operator $(\tilde{\mathbf{I}} - \tilde{\mathbf{a}}''\mathscr{W}/n)^{-1}$ reinstates radiation losses corresponding to fluctuation scattering governed by the packing factor $\mathscr{W}$.

To illustrate the above we start with Rayleigh's results[15] for elliptic cylinders and ellipsoids in the form

$$a'_0 \approx i\Gamma\mathscr{C}, \quad \mathscr{C} = C - 1;$$
$$a'_i = -i\Gamma\beta/(1 + \beta q_i) \equiv -i\Gamma\mathfrak{B}'_i, \quad \mathscr{B} = B - 1;$$
$$\Gamma = K^2\mathscr{V}/4, \quad k^3\mathscr{V}/4\pi; \quad \operatorname{Im} C>0, \operatorname{Im} B<0. \qquad (112)$$

Here $\mathscr{V}$ is the particle's volume, C and B are its relative parameters (compressibility and inverse mass density in the simplest cases), and q_i is the elliptical depolarization integral[16] normalized to $\Sigma q_i = 1$; in two dimensions, in terms of the principal diameters d_1 and d_2 along z and x, $q_1 = 1 - q_2 = d_2/(d_1 + d_2)$. For rigid scatterers, $C = B = 0$, and $a'_0 \approx -i\Gamma$, $a'_i \approx i\Gamma/(1 - q_i)$. More generally than in (112), we replace B by a tensor parameter $\tilde{\mathbf{B}} = (B_{ij})$.

If the principal axes are along the Cartesian axes, then for cylindrical dipoles, in terms of I_2 of (70),

$$\mathscr{D}_i = 1 + \mathfrak{B}'_i\epsilon_i, \quad \epsilon_1 = \epsilon = \mathscr{V}I_2/2\pi b^2, \quad \epsilon_2 = -\epsilon. \qquad (113)$$

For ellipsoids, in terms of $I_2M_2^0$ and $I_2M_2^2$ of (87),

$$\epsilon_1 = \epsilon, \quad \epsilon_2 = -\tfrac{1}{2}(\epsilon + \epsilon'), \quad \epsilon_3 = -\tfrac{1}{2}(\epsilon - \epsilon');$$
$$\epsilon = \mathscr{V}I_2M_2^0/4\pi b_2^3, \quad \epsilon' = 3\mathscr{V}I_2M_2^2/4\pi b_2^3 \qquad (114)$$

for both cases, $\Sigma\epsilon_i = 0$, and consequently a''_i has the same form as a'_i,

$$a''_i = \frac{a'_i}{\mathscr{D}_i} = \frac{-i\Gamma\mathscr{B}_i}{1 + \mathscr{B}_iQ_i} \equiv -i\Gamma\mathfrak{B}_i,$$
$$Q_i = q_i + \epsilon_i, \quad \Sigma Q_i = 1, \qquad (115)$$

such that the packing effects are incorporated in new depolarization factors Q_i. The relation $a''_0 = a'_0 = i\Gamma\mathscr{C}$ plus Eq. (115) determine an ellipsoid having the same volume and physical parameters as the original, but a different shape. The equivalent ellipsoid is flatter along the array normal and broader in the plane of centers, the elongation being greater along the smaller diameter of the exclusion ellipse (along b_2 for $b_2 < b_3$). Corresponding to (111), we have

$$\mathscr{g}_{r0} \approx i\Gamma[\mathscr{C} - \Sigma\mathfrak{B}_i\gamma_i\gamma_i^0]$$
$$- \mathscr{W}\Gamma^2(\mathscr{C}^2 + \Sigma\mathfrak{B}_i^2\gamma_i\gamma_i^0/n), \qquad (116)$$

where we keep only the leading terms in $\mathscr{W}\Gamma^2$, and use $\mathscr{C}^2 \approx (\operatorname{Re}\mathscr{C})^2 \approx |\mathscr{C}|^2$, etc. Substituting (116) into (50) and (51) we generalize the earlier results by replacing $\mathscr{F}$ corresponding to g' by the present factor for g'' and thereby include the dominant multipole coupling effects.

More generally, for an ellipsoid of arbitrary orientation with shape dyadic $\tilde{\mathbf{q}} = \Sigma q_i\hat{\boldsymbol{\xi}}_i\hat{\boldsymbol{\xi}}_i$ and dyadic parameter $\tilde{\mathbf{B}} = \Sigma B_i\boldsymbol{\eta}_i\boldsymbol{\eta}_i$, we have

$$\tilde{\mathbf{a}}' = -i\Gamma(I + \tilde{\mathscr{B}}\cdot\tilde{\mathbf{q}})^{-1}\cdot\tilde{\mathscr{B}} \equiv -i\Gamma\tilde{\mathfrak{B}}', \quad \tilde{\mathscr{B}} = \tilde{\mathbf{B}} - \tilde{\mathbf{I}}. \qquad (117)$$

Writing the array coupling factors ϵ_i with $\Sigma\epsilon_i = 0$ as the array dyadic $\tilde{\boldsymbol{\epsilon}} = \Sigma\epsilon_i\hat{\mathbf{z}}_i\hat{\mathbf{z}}_i$ (where $\hat{\mathbf{z}}_i = \hat{\mathbf{z}},\hat{\mathbf{x}},\hat{\mathbf{y}}$), we use

$-i\tilde{a}'\tilde{\mathcal{N}}=\tilde{\mathfrak{B}}'\cdot\tilde{\epsilon}$, and $(\tilde{I}-i\tilde{a}'\tilde{\mathcal{N}})^{-1}\cdot\tilde{a}'=(\tilde{I}+\tilde{\mathfrak{B}}'\cdot\tilde{\epsilon})^{-1}\cdot\tilde{a}'$ to obtain

$$\tilde{a}''=-i\Gamma(\tilde{I}+\tilde{\mathscr{B}}\cdot\tilde{Q})^{-1}\cdot\tilde{\mathscr{B}}\equiv-i\Gamma\tilde{\mathfrak{B}},\quad \tilde{Q}=\tilde{q}+\tilde{\epsilon};$$
$$\tilde{a}=-i\Gamma\tilde{\mathfrak{B}}-(\mathscr{W}\Gamma^2/n)\tilde{\mathfrak{B}}\cdot\tilde{\mathfrak{B}},\tag{118}$$

with Q as the net depolarization dyadic. The corresponding equivalent scattering amplitude, obtained by substituting (118) in (111),

$$g_{r0}\approx i\Gamma(\mathscr{C}-\hat{r}\cdot\tilde{\mathfrak{B}}\cdot\hat{k})-\mathscr{W}\Gamma^2(\mathscr{C}^2+\hat{r}\cdot\tilde{\mathfrak{B}}\cdot\tilde{\mathfrak{B}}\cdot\hat{k}/n),\tag{119}$$

as well as the special case (116), can be used in (44) ff to construct T, R, and $|G_{r0}|^2$ explicitly.

We apply (119) to illustrate statements for (52) ff with reference to theorems for $T=T_{00}$, $T'=T_{0'0'}$, etc. It suffices to consider coincident principal axes for the parameter $(\tilde{B})$ and shape $(\tilde{q})$ of the particle, and particle orientation specified by $\hat{\xi}_3=\hat{y},\hat{\xi}_1=\hat{z}\cos\beta-\hat{x}\sin\beta,\hat{\xi}_2=\hat{y}\times\hat{\xi}_1$. For incidence and observation in the zy plane, we require only

$$\tilde{\mathfrak{B}}=(\tilde{I}+\tilde{\mathfrak{B}}'\cdot\tilde{\epsilon})^{-1}\cdot\tilde{\mathfrak{B}}'=[(\mathfrak{B}'_{11}+\mathfrak{B}'_1\mathfrak{B}'_2\epsilon_2)\hat{z}\hat{z}$$
$$+\mathfrak{B}'_{12}(\hat{z}\hat{x}+\hat{x}\hat{z})+(\mathfrak{B}'_{22}+\mathfrak{B}'_1\mathfrak{B}'_2\epsilon_1)\hat{x}\hat{x}]/D,$$
$$D=1+\mathfrak{B}'_{11}\epsilon_1+\mathfrak{B}'_{22}\epsilon_2+\mathfrak{B}'_1\mathfrak{B}'_2\epsilon_1\epsilon_2,$$
$$\mathfrak{B}'_i=\mathscr{B}_i(1+\mathscr{B}_iq_i)^{-1},\quad \mathscr{B}_i=B_i-1,\tag{120}$$

where

$$\mathfrak{B}'_{11}=\mathfrak{B}'_1\cos^2\beta+\mathfrak{B}'_2\sin^2\beta,$$
$$\mathfrak{B}'_{22}=\mathfrak{B}'_1\sin^2\beta+\mathfrak{B}'_2\cos^2\beta,$$
$$\mathfrak{B}'_{12}=\mathfrak{B}'_{21}=(\mathfrak{B}'_2-\mathfrak{B}'_1)\sin\beta\cos\beta.\tag{121}$$

Equivalently, in terms of $\tilde{Q}=\tilde{q}+\tilde{\epsilon}=(Q_{ij})=(q_{ij}+\epsilon_i\delta_{ij})$ with q_{ij} expressed in terms of q_1 and q_2 by the relations in (121),we obtain from the form $\tilde{\mathfrak{B}}=(\tilde{I}+\tilde{\mathscr{B}}\cdot\tilde{Q})^{-1}\cdot\tilde{\mathscr{B}}$,

$$\tilde{\mathfrak{B}}=[(\mathscr{B}_{11}+\mathscr{B}_1\mathscr{B}_2Q_{22})\hat{z}\hat{z}+(\mathscr{B}_{21}-\mathscr{B}_1\mathscr{B}_2Q_{21})$$
$$\times(\hat{z}\hat{x}+\hat{x}\hat{z})+(\mathscr{B}_{22}+\mathscr{B}_1\mathscr{B}_2Q_{11})\hat{x}\hat{x}]/D',$$
$$D'=1+\mathscr{B}_{11}Q_{11}+\mathscr{B}_{22}Q_{22}+2\mathscr{B}_{12}Q_{12}$$
$$+\mathscr{B}_1\mathscr{B}_2(Q_{11}Q_{22}-Q_{12}^2),\tag{122}$$

with $\mathscr{B}_{ij}$ in terms of $\mathscr{B}_i$ as in (121). From (120) or (122) with $\tilde{\mathfrak{B}}=(\mathfrak{B}_{ij})$, we have

$$\tilde{\mathfrak{B}}\cdot\tilde{\mathfrak{B}}=(\mathfrak{B}_{11}^2+\mathfrak{B}_{12}^2)\hat{z}\hat{z}+\mathfrak{B}_{12}(\mathfrak{B}_{11}+\mathfrak{B}_{22})$$
$$\times(\hat{z}\hat{x}+\hat{x}\hat{z})+(\mathfrak{B}_{22}^2+\mathfrak{B}_{12}^2)\hat{x}\hat{x},\tag{123}$$

for which we use $(\mathfrak{B}_{ij})^2\approx(\mathrm{Re}\,\mathfrak{B}_{ij})^2$.

Corresponding to the generalization of C_{g} in (51), we write $\mathscr{F}=\mathscr{R}+i\mathscr{J}$ with $\mathscr{R}$ and $\mathscr{J}$ in terms of the present results. Thus, for absorption of the order of the scattering losses,

$$\mathrm{Im}\,C_{g21}=K\mathscr{R}_{21},\quad \mathrm{Re}\,C_{g21}=K(\mathscr{J}_{21}+\Gamma\mathscr{W}\mathscr{M}\mathscr{R}_{2r}\mathscr{R}_{r0}),$$
$$K=\rho\mathscr{V}k/4\gamma_0=C\Gamma,$$
$$\mathscr{R}_{21}=\mathrm{Re}(\mathscr{C}-\hat{r}_2\cdot\tilde{\mathfrak{B}}\cdot\hat{r}_1),\quad \mathscr{J}_{21}=\mathrm{Im}(\mathscr{C}+\hat{r}_2\cdot\tilde{\mathfrak{B}}\cdot\hat{r}_1),$$
$$\mathscr{M}\mathscr{R}_{2r}\mathscr{R}_{r0}=(\mathrm{Re}\,\mathscr{C})^2+\mathrm{Re}(\hat{r}_2\cdot\tilde{\mathfrak{B}})\cdot\mathrm{Re}\,\tilde{\mathfrak{B}}\cdot\hat{r}_1/n.\tag{124}$$

The angle dependent terms for $\hat{r}_i=\hat{z}\cos\theta_i+\hat{x}\sin\theta_i=\hat{z}\gamma_i+\hat{x}\alpha_i$ are given by

$$\hat{r}_2\cdot\tilde{\mathfrak{B}}\cdot\hat{r}_1=\mathfrak{B}_{11}\gamma_2\gamma_1+\mathfrak{B}_{12}(\gamma_2\alpha_1+\alpha_2\gamma_1)+\mathfrak{B}_{22}\alpha_2\alpha_1,$$
$$\hat{r}_2\cdot\tilde{\mathfrak{B}}\cdot\tilde{\mathfrak{B}}\cdot\hat{r}_1=(\mathfrak{B}_{11}^2+\mathfrak{B}_{12}^2)\gamma_2\gamma_1+\mathfrak{B}_{12}(\mathfrak{B}_{11}+\mathfrak{B}_{22})$$
$$\times(\gamma_2\alpha_1+\alpha_2\gamma_1)+(\mathfrak{B}_{22}^2+\mathfrak{B}_{12}^2)\alpha_2\alpha_1.\tag{125}$$

Specializing to $\hat{r}_2,\hat{r}_1=\hat{k}',\hat{k}$ and $\hat{k},\hat{k}'$, in terms of $\hat{k}=\hat{z}\gamma+\hat{x}\alpha$ and $\hat{k}'=\hat{z}\gamma+\hat{x}\alpha$, we have

$$\hat{k}'\cdot\tilde{\mathfrak{B}}\cdot\hat{k}=\hat{k}\cdot\tilde{\mathfrak{B}}\cdot\hat{k}'=-\mathfrak{B}_{11}\gamma^2+\mathfrak{B}_{22}\alpha^2,$$
$$\hat{k}'\cdot\tilde{\mathfrak{B}}\cdot\tilde{\mathfrak{B}}\cdot\hat{k}=\hat{k}\cdot\tilde{\mathfrak{B}}\cdot\tilde{\mathfrak{B}}\cdot\hat{k}'$$
$$=-(\mathfrak{B}_{11}^2+\mathfrak{B}_{12}^2)\gamma^2+(\mathfrak{B}_{22}^2+\mathfrak{B}_{12}^2)\alpha^2.\tag{126}$$

Consequently, from $R_{21}=2Cg_{21}/D$ of (42)

$$\frac{g_{0'0}}{g_{00'}}=\frac{R_{0'0}}{R_{00'}}=\frac{R}{R'}=1.\tag{127}$$

Thus, although the particle alignment is not symmetrical to the array, the reflection coefficients for arbitrary imaged directions $\hat{k}$ and $\hat{k}'$ are identical in magnitude and phase.

On the other hand if $\hat{r}_2$ and $\hat{r}_1$ both equal $\hat{k}$ or $\hat{k}'$, then

$$\hat{k}\cdot\tilde{\mathfrak{B}}\cdot\hat{k}=\mathfrak{B}_{11}\gamma^2+\mathfrak{B}_{12}2\gamma\alpha+\mathfrak{B}_{22}\alpha^2,$$
$$\hat{k}'\cdot\tilde{\mathfrak{B}}\cdot\hat{k}'=\mathfrak{B}_{11}\gamma^2-\mathfrak{B}_{12}2\gamma\alpha+\mathfrak{B}_{22}\alpha^2,\tag{128}$$

and similarly for the $\tilde{\mathfrak{B}}\cdot\tilde{\mathfrak{B}}$ terms. Thus, from (42) for $T_{00}=T$ and $T_{0'0'}=T'$,

$$DT-(1-C^2\|g\|)=-[DT'-(1-C^2\|g\|)]$$
$$=C(g_{00}-g_{0'0'})$$
$$=iK4\mathfrak{B}_{12}\gamma\alpha-K4\Gamma\mathscr{W}\,\mathrm{Re}\,\mathfrak{B}_{12}$$
$$\times\mathrm{Re}\,(\mathfrak{B}_{11}+\mathfrak{B}_{22})\gamma\alpha,\tag{129}$$

so that T and T' differ in general. However, for negligible losses, $g_{12}\approx i\mathrm{Im}g_{12}$, and $\|g\|=g_{00}g_{0'0'}-g_{00'}^2$ is real; for such cases, (129) in terms of real quantities A and B reduces to

$$T=\frac{A+iB}{D},\quad T'=\frac{A-iB}{D},\quad |T|^2=|T'|^2=\frac{A^2+B^2}{|D|^2},\tag{130}$$

so that the transmitted magnitudes are equal for $\hat{k}$ and $\hat{k}'$. This plus (127) illustrates the general considerations of magnitudes after (54), and (126)–(129) serve for the related discussion of the phases, i.e., that the sum of the reflected phases $(\Theta_R+\Theta_{R'}=2\Theta_R)$ and the sum of the transmitted phases $(\Theta_T+\Theta_{T'})$ differ by π for negligible losses. From (129), the transmitted phases differ by $\Theta_T-\Theta_{T'}\approx 8K\,\mathrm{Re}\,\mathfrak{B}_{12}\gamma\alpha$ to first approximation.

The development of (112) excludes pressure release obstacles for which a'_0 is not of order k^n, and the case of monopole resonance. We include these by an alternative form for the monopole $\mathscr{C}\gg 1$ (corresponding, e.g., to underwater sound incident on gaseous cylinders or bubbles),

$$a'_0\approx\frac{i\Gamma\mathscr{C}}{1-\Gamma\mathscr{C}\mathscr{L}};\quad \mathscr{L}=\frac{2\ell_0}{\pi},\frac{1}{k\ell},\tag{131}$$

where $\ell_0=\ln[8/kc(d_1+d_2)]$, and ℓ is the electrostatic capacity of the ellipsoid. For the pressure release case, we let $C\to\infty$ in $\mathscr{C}=C-1$ of (131), and $B\to\infty$ in a'_i of (112) to obtain

$$a_0' \approx \frac{-i}{\mathscr{L}} = \frac{-i\pi}{2\ell_0}, \quad -ik\ell; \quad a_i' \approx -\frac{i\Gamma}{q_i} = \mathscr{O}(k^n), \tag{132}$$

where the monopole dominates for small k. The form in (131) also suffices near the first resonance for large finite $\mathscr{C} \approx C$. The resonant frequencies correspond to $\Gamma\mathscr{C}\mathscr{L} = 1$, i.e.,

$$k_{0\wedge}^2 \approx 2\pi/\mathscr{V}\mathscr{C}\ell_0, \quad 4\pi\ell/\mathscr{V}\mathscr{C}, \tag{133}$$

for which the isolated monopole $a_0 = a_0'/(1-a_0')$ reduces to $a_\wedge = -1$.

Corresponding to (132) or (133), we neglect the dipole and use

$$g \approx a_0 \approx a_0'/(1 - a_0' i\mathscr{N}_{00} - a_0'\mathscr{W}) = a_0''/(1 - a_0''\mathscr{W}),$$
$$a_0'' = a_0'/(1 - a_0' i\mathscr{N}_{00}), \tag{134}$$

in terms of $\mathscr{N}_{00} = \mathscr{N}_0 = \mathfrak{R}_0$ of (69) or $\mathscr{N}_0^0 = \mathfrak{R}_0^0/kb_2$ of (86). The corresponding transmissions and reflection coefficients satisfy

$$T = \frac{1}{1-2Cg} = 1 + \frac{2Cg}{1-2Cg} = 1 + R,$$
$$|T|^2 = 1 - |R|^2 + \frac{4C\,\mathrm{Re}\,g}{|1-2Cg|^2}, \tag{135}$$

where the last term equals $-\rho\mathscr{S}\sec\theta_0$ as in (44). Near resonance, we write

$$a_0'' = i\Gamma\mathscr{C}/[1 - \Gamma\mathscr{C}(\mathscr{L} - \mathscr{N}_{00})] = i\Gamma\mathscr{C}/(1 - k^2/k_\wedge^2),$$
$$k_\wedge^2 = \frac{k_{0\wedge}^2}{(1-x)} > k_{0\wedge}^2; \quad x = \frac{|I^0|}{\ell_0}, \quad \frac{|I_0| M_0^0 \ell}{b_2}, \tag{136}$$

so that multipole coupling shifts the resonances to higher frequencies. At resonance, $g_{0\wedge} = -1/\mathscr{W}$ with $|g_{0\wedge}| > |a_{0\wedge}|$. Consequently

$$T_\wedge = \frac{\mathscr{W}}{2C + \mathscr{W}}, \quad R_\wedge = \frac{-2C}{2C+\mathscr{W}} = -1 + \frac{\mathscr{W}}{2C+\mathscr{W}},$$
$$\rho\mathscr{S}_\wedge \sec\theta_0 = \frac{4C\mathscr{W}}{|2C+\mathscr{W}|^2}, \tag{137}$$

and $R_\wedge$ dominates. With increasing w in (48), $\mathscr{W}$ becomes small (zero in the periodic limits), and $T_\wedge \approx 0$, $R_\wedge \approx -1$, and $\mathscr{S}_\wedge \approx 0$; thus such bubble screens approximate perfect reflectors.

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